On some special classes of contact B_0-VPG graphs

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Abstract

A graph G is a B_0-VPG graph if one can associate a horizontal or vertical path on a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect in at least one grid-point. A graph G is a *contact B_0-VPG graph* if it is a B_0-VPG graph admitting a representation with no one-point paths, no two paths crossing, and no two paths sharing an edge of the grid. In this paper, we present a minimal forbidden induced subgraph characterisation of contact B_0-VPG graphs within four special graph classes: chordal graphs, tree-cographs, P_4-tidy graphs and P_5-free graphs. Moreover, we present a polynomial-time algorithm for recognising chordal contact B_0-VPG graphs.

Keywords: contact B_0-VPG graph, chordal graph, tree-cograph, P_4-tidy graph, P_5-free graph.

1. Introduction

Golumbic et al. introduced in [2] the concept of *vertex intersection graphs of paths in a grid* (referred to as *VPG graphs*). An undirected graph $G = (V, E)$ is called a VPG graph if one can associate a path in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect in at least one grid-point. In the seminal paper on VPG graphs it was shown that this class is equivalent to the earlier defined class of string graphs [14].

Under the perspective of paths in grids, a particular attention was paid to the case where the paths have a limited number of bends. An undirected graph

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$G = (V, E)$ is then called a B_k-*VPG graph*, for some integer $k \geq 0$, if one can associate a path with at most k bends in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect in at least one grid-point. Recognition of VPG graphs is NP-complete by the equivalence with string graphs. Moreover B_k-VPG recognition is NP-complete for all k [6].

Since their introduction, B_k-VPG graphs have been studied by many researchers and the community of people working on these graph classes or related ones is still growing (see for instance [1, 2, 5, 7, 8, 9, 15, 18]).

In this paper, we are interested in a subclass of B_k-VPG graphs called *contact* B_k-*VPG*. A *contact* B_k-*VPG representation* of G is a VPG representation in which each path has length at least one, at most k bends, and intersecting paths neither cross each other nor share an edge of the grid. A graph is a *contact* B_k-*VPG graph* if it has a contact B_k-VPG representation. Here, we will focus on the special case when $k = 0$, i.e. each path is a horizontal or vertical path in the grid.

Contact graphs in general (graphs where vertices represent geometric objects which are allowed to touch but not to cross each other, a natural model arising from real physical objects) have been considered in the past (see for instance [10, 11, 19, 20]). In particular, for intersection models of lines in the plane, it is often the case that three lines intersecting at a same point is not allowed, but we do not impose such a restriction.

As for many graph classes having not many known full characterisations (for example, a complete list of minimal forbidden induced subgraphs is not known), their characterisation within well studied graphs classes or with respect to graph parameters was investigated. In the case of contact B_k-VPG graphs, it was shown in [12] that every planar bipartite graph is a contact B_0-VPG graph. Later, in [7], the authors show that every triangle-free planar graph is a contact B_1-VPG graph. In a recent paper (see [13]), contact B_k-VPG graphs have been investigated from a structural point of view and it was for instance shown that they do not contain cliques of size 7 and they always contain a vertex of degree at most 6. Moreover, it was shown that they are 6-colourable. Regarding contact B_0-VPG graphs, it was shown that they are 4-colourable. Furthermore, 3-colouring and the recognition problem were shown to be NP-complete.

In this paper, our goal is to get a better understanding and knowledge of the underlying structure of contact B_0-VPG graphs. Even though classical graph problems may be difficult to solve in this graph class (see for instance [13]), a better knowledge of its structural properties may lead to good approximation algorithms for these problems. We will consider the following four special graph classes: chordal graphs, tree-cographs, P_4-tidy graphs and P_5-free graphs, and we will characterise those graphs from these families that are contact B_0-VPG. Moreover, we will present a polynomial-time algorithm for recognising chordal contact B_0-VPG graphs based on our characterisation. For the other graph classes considered here, the characterisation immediately yields a polynomial-time recognition algorithm.

A preliminary version of this paper appears in [4].

2. Preliminaries

For concepts and notations not defined here we refer the reader to [3]. All graphs in this paper are simple (i.e., without loops or multiple edges). Let $G = (V, E)$ be a graph. If $u, v \in V$ and $uv \notin E$, uv is called a *nonedge* of G. We write $G - v$ for the subgraph obtained by deleting a vertex v and all the edges incident to v. Similarly, we write $G - e$ for the subgraph obtained by deleting an edge e without deleting its endpoints.

For each vertex v of G, $N_G(v)$ denotes the *neighbourhood* of v in G and $N_G[v]$ denotes the *closed neighbourhood*, i.e. $N_G(v) \cup \{v\}$. For a set $A \subseteq V$, we denote by $N(A)$ the set of vertices having a neighbour in A, and by $N[A]$ the set of vertices belonging to A or having a neighbour in A. Two vertices v and w of G are *false twins* (resp. *true twins*) if $N_G(v) = N_G(w)$ (resp. $N_G[v] = N_G[w]$).

Given a subset $A \subseteq V$, $G[A]$ stands for *the subgraph of G induced by A*, and $G \setminus A$ denotes the *induced subgraph* $G[V \setminus A]$. We say that a vertex $v \in V \setminus A$ is *complete to A* if v is adjacent to every vertex of A, and that v is *anticomplete to A* if v has no neighbour in A. Similarly, we say that two disjoint sets $A, B \subset V$ are *complete* (resp. *anticomplete*) to each other if every vertex in A is complete (resp. anticomplete) to B.

A *clique* is a set of pairwise adjacent vertices. A vertex v is *simplicial*, if $N_G(v)$ is a clique. A *stable set* is a set of vertices no two of which are adjacent. A *complete graph* is a graph such that all its vertices are adjacent to each other, i.e. a graph induced by a clique. The *complete graph* on n vertices is denoted by K_n. In particular, K_3 is called a *triangle*. K_4^- stands for the graph obtained from K_4 by deleting exactly one edge.

The *complement graph* of $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ such that $\overline{E} = \{uv|\ uv \notin E\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *disjoint union of G_1 and G_2*, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2$. The *join of G_1 and G_2*, denoted by $G_1 \vee G_2$, is the graph obtained by first taking the disjoint union of G_1 and G_2 and then making V_1 and V_2 complete to each other. Notice that $\overline{G_1 \cup G_2} = \overline{G_1} \vee \overline{G_2}$.

Given a graph H, we say that G *contains no induced H*, if G contains no induced subgraph isomorphic to H. If $\mathcal{H}$ is a family of graphs, G is said to be *$\mathcal{H}$-free* if G contains no induced subgraph isomorphic to some graph belonging to $\mathcal{H}$.

Let $\mathcal{G}$ be a class of graphs. A graph belonging to $\mathcal{G}$ is called a *$\mathcal{G}$-graph*. If $G \in \mathcal{G}$ implies that every induced subgraph of G is a $\mathcal{G}$-graph, $\mathcal{G}$ is said to be *hereditary*. If $\mathcal{G}$ is a hereditary class, a graph H is a *minimal forbidden induced subgraph of $\mathcal{G}$*, or more briefly, *minimally non-$\mathcal{G}$*, if H does not belong to $\mathcal{G}$ but every proper induced subgraph of H is a $\mathcal{G}$-graph.

A *path* is a sequence of vertices $v_1, \ldots, v_k$ such that v_i is adjacent to v_{i+1}, for $i = 1, \ldots, k-1$. The vertices $v_2, \ldots, v_{k-1}$ are called *internal vertices* of the path. If there is no edge v_iv_j such that $|i - j| \geq 2$, the path is said to be *chordless* or *induced*. A *cycle* C is a sequence of vertices $v_1, \ldots, v_k$ such that v_i is adjacent to v_{i+1} for $i = 1, \ldots, k$, where indices are taken modulo k. If there is no edge v_iv_j such that $|i - j| \geq 2$, C is said to be *chordless* or *induced*. The

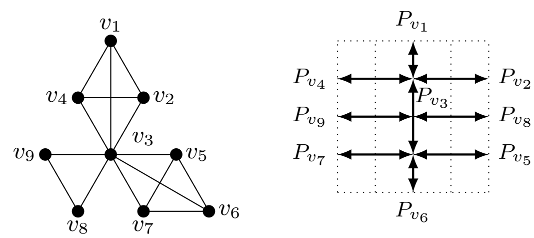


Figure 1: A graph G and a contact B_0-VPG representation of it.

induced path (resp. induced cycle) on n vertices is denoted P_n (resp. C_n). A graph is called *chordal* if it does not contain any chordless cycle of length at least four. A *block graph* is a chordal graph which is K_4^--free.

A graph is *bipartite*, if its vertex set can be partitioned into two stable sets. If, in addition, the two stable sets are complete to each other, the graph is called *complete bipartite*. $K_{n,m}$ stands for the complete bipartite graph whose vertex set can be partitioned into two stable sets V_1, V_2 such that $|V_1| = n$ and $|V_2| = m$.

A graph G is *connected*, if for each pair of vertices u, v there exists a path from u to v. A *tree* is a connected graph with no induced cycle. Given a connected graph $G = (V, E)$, the *distance between two vertices* $u, v \in V$, denoted by $d_G(u, v)$, is the number of edges of a shortest path from u to v. The *diameter* of G is the maximum distance between two vertices.

An undirected graph $G = (V, E)$ is called a *B_k-VPG graph*, for some integer $k \geq 0$, if one can associate a path with at most *k bends* (a bend is a 90 degrees turn of a path at a grid-point) on a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect in at least one grid-point. Such a representation is called a *B_k-VPG representation*. The horizontal grid lines will be referred to as *rows* and denoted by $x_0, x_1, \ldots$ and the vertical grid lines will be referred to as *columns* and denoted by $y_0, y_1, \ldots$. We are interested in a subclass of B_0-VPG graphs called contact B_0-VPG. A *contact B_0-VPG representation* $\mathcal{R}(G)$ of G is a B_0-VPG representation in which each path in the representation is either a horizontal path or a vertical path on the grid, with length at least one (the length is the number of grid-points minus one), such that two vertices are adjacent if and only if the corresponding paths intersect in at least one grid-point without crossing each other and without sharing an edge of the grid. A graph is a *contact B_0-VPG graph* if it has a contact B_0-VPG representation. For every vertex v, we denote by P_v the corresponding path in $\mathcal{R}(G)$ (see Figure 1). Consider a clique K in G. A path P_v representing a vertex $v \in K$ is called a *path of the clique K*.

Let us start with an easy but very helpful lemma.

Lemma 1. *Let G be a contact B_0-VPG graph. Then the size of a biggest clique* *in G is at most 4, i.e. G is K_5-free.*

PROOF. Given two adjacent vertices in G, the intersection of their paths in any contact B_0-VPG representation is exactly one grid point. Moreover, it is easy to see that all paths corresponding to vertices in a clique of G must intersect in the same grid point. Assume there is a clique K of size 5 in G and let P be the point of intersection of the corresponding paths in the grid. At least two of the paths must be in the same row or the same column, and contain at least one grid edge intersecting P (a path cannot be only a grid point), a contradiction. □

Remark 2. *Let G be a K_4^--free graph containing an induced cycle C of at least 4 vertices. Then no vertex is adjacent to 3 consecutive vertices of C.*

Let G be a contact B_0-VPG graph, and K be a clique in G. A vertex v is called an *end* in a contact B_0-VPG representation of K if the grid point representing the intersection of the paths of K corresponds to an endpoint of P_v.

Remark 3. *Let G be a contact B_0-VPG graph, and K be a clique in G of size four. Then, every vertex in K is an end in any contact B_0-VPG representation of K.*

Lemma 4. *In any contact B_0-VPG representation of C_4, the union of the paths representing vertices in C must enclose a rectangle of the grid.*

PROOF. Consider a B_0-VPG representation of C_4. At least two vertices, say a and b, in C have the same direction. We can assume that P_a and P_b are both vertical. If a and b are adjacent, then the corresponding paths intersect in a row x_i of the grid. One of them, say P_a, is above x_i and the other is below x_i. Let c be the vertex adjacent to a and non adjacent to b. Clearly, the path P_c representing c must be also above x_i. Similarly, the path representing the vertex d adjacent to b and non adjacent to a must be below x_i. But then it is impossible for P_c and P_d to intersect. Therefore, a and b are non adjacent. Now, it is clear that P_c and P_d must be both horizontal, otherwise we could repeat the previous argument. If P_a and P_b lie in columns y_i and y_j, then P_c and P_d must contain all points of the grid between y_i and y_j in their respective columns, say x_k and x_l. Then, these paths enclose the rectangle limited by rows y_i, y_j and columns x_k, x_l. □

In what follows, we give a set of graphs that are not contact B_0-VPG graphs. We will use this result later to obtain our characterisations. Let H_0 denote the graph composed of three K_4's that share a common vertex and such that there are no further edges (see Figure 2).

Lemma 5. *If G is a contact B_0-VPG graph, then G is $\{K_5, K_{3,3}, H_0, K_4^-\}$-free.*

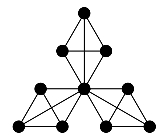


Figure 2: The graph H_0.

PROOF. Let G be a contact B_0-VPG graph. It immediately follows from Lemma 1 that G is K_5-free.

Now consider the graph $K_{3,3}$. Let C be a cycle of length four in $K_{3,3}$ induced by the vertices a, b, c, d. If $K_{3,3}$ is contact B_0-VPG, then, by Lemma 4, in any contact B_0-VPG representation of C, the union of the paths representing vertices in C must enclose a rectangle of the grid. Assume that P_a, P_c are horizontal paths, and P_b, P_d are vertical paths. Now, consider vertices e and f in $K_{3,3}$ with e being adjacent to a and c, and f being adjacent to b and d. Each of the paths P_e, P_f must intersect opposite paths of the rectangle. Clearly, P_e must be a vertical path and P_f must be a horizontal. If P_e is contained inside the rectangle, then it is impossible for P_f to intersect P_b, P_d while being inside the rectangle without crossing P_e. So P_f must be outside the rectangle, but then it cannot intersect P_e. If P_e lies outside the rectangle, then of course P_f has to lie outside the rectangle as well, otherwise it cannot intersect P_e. But now it cannot intersect both P_b, P_d without crossing at least one of them. So we conclude that $K_{3,3}$ is not B_0-VPG.

Now let v, w be two adjacent vertices in G. Then, in any contact B_0-VPG representation of G, P_v and P_w intersect at a grid-point P. Clearly, every common neighbour of v and w must also contain P. Hence, v and w cannot have two common neighbours that are non-adjacent. So, G is K_4^--free.

Finally, consider the graph H_0 which consists of three cliques of size four, say A, B and C, with a common vertex x. Suppose that H_0 is contact B_0-VPG. Then, it follows from Remark 3 that every vertex in H_0 is an end in any contact B_0-VPG representation of H_0. In particular, vertex x is an end in any contact B_0-VPG representation of A, B and C. In other words, the grid-point representing the intersection of the paths of each of these three cliques corresponds to an endpoint of P_x. Since these cliques have only vertex x in common, these grid-points are all distinct. But this is a contradiction, since P_x has only two endpoints. So we conclude that H_0 is not contact B_0-VPG, and hence the result follows. □

3. Chordal graphs

In this section, we will consider chordal graphs and characterise those that are contact B_0-VPG. First, let us point out the following corollary.

Corollary 6. *A chordal contact B_0-VPG graph is a block graph.*

This follows directly from Lemma 5 and the definition of block graphs.

The following lemma states an important property of minimal chordal non contact B_0-VPG graphs that contain neither K_5 nor K_4^-.

Lemma 7. *Let G be a $\{K_5, K_4^-\}$-free graph. If G is a minimal non contact B_0-VPG graph, then every simplicial vertex of G has degree exactly three.*

Proof. Since G is K_5-free, every clique in G has size at most four. Therefore, every simplicial vertex has degree at most three. Let v be a simplicial vertex of G. Assume first that v has degree one and consider a contact B_0-VPG representation of $G - v$ (which exists since G is minimal non contact B_0-VPG). Let w be the unique neighbour of v in G. Without loss of generality, we may assume that the path P_w lies on some row of the grid. Now clearly, we can add one extra column to the grid between any two consecutive vertices of the grid belonging to P_w and adapt all paths without changing the intersections (if the new column is added between column y_i and y_{i+1}, we extend all paths containing a grid-edge with endpoints in column y_i and y_{i+1} in such a way that they contain the new edges in the same row and between column y_i and y_{i+2} of the new grid, and any other path remains the same). But then we may add a path representing v on this column which only intersects P_w (adding a row to the grid and adapting the paths again, if necessary) and thus, we obtain a contact B_0-VPG representation of G, a contradiction. So suppose now that v has degree two, and again consider a contact B_0-VPG representation of $G - v$. Let w_1, w_2 be the two neighbours of v in G. Then, w_1, w_2 do not have any other common neighbour since G is K_4^--free. Let P be the grid-point corresponding to the intersection of the paths P_{w_1} and P_{w_2}. Since these paths do not cross and since w_1, w_2 do not have any other common neighbour (except v), there is at least one grid-edge having P as one of its endpoints and which is not used by any path of the representation. But then we may add a path representing v by using only this particular grid-edge (or adding a row/column to the grid that subdivides this edge and adapting the paths, if the other endpoint of the grid-edge belongs to a path in the representation). Thus, we obtain a contact B_0-VPG representation of G, a contradiction. We conclude therefore that v has degree exactly three. □

Let v be a vertex of a contact B_0-VPG graph G. An endpoint of its corresponding path P_v is *free* in a representation of G, if P_v does not intersect any other path at that endpoint; v is called *internal* if no representation of G with a free endpoint of P_v exists. If in a representation of G a path P_v intersects a path P_w but not at an endpoint of P_w, v is called a *middle neighbour* of w.

In the following two lemmas we associate the fact of being or not an internal vertex of G with the contact B_0-VPG representation of G.

Lemma 8. *Let G be a chordal contact B_0-VPG graph and let v be a non internal vertex in G. Then, there exists a contact B_0-VPG representation of G in which all the paths representing vertices in $G - v$ lie to the left of a free endpoint of P_v (by considering P_v as a horizontal path).*

Proof. We will do a proof by induction on the number of vertices of G. If there is only one vertex in G the result is trivial. Suppose G is a graph with at least two vertices. Consider a contact B_0-VPG representation of G. Without loss of generality, we may assume that P_v lies on a row x_i between columns y_j, y_k, $j < k$, and its right endpoint is free. Such a representation exists, since v is not internal.

If v is a middle neighbour of another vertex, say u, we do the following. Assume P_u lies on column y_j between rows x_ℓ and x_t, $\ell < t$. We split P_u into two paths, P_{u_1}, P_{u_2}, such that P_{u_1} goes from row x_i to row x_t and P_{u_2} goes from row x_ℓ to row x_i (see Figure 3). We denote the corresponding graph by G^*. If v is not a middle vertex of another vertex, then we simply set $G^* = G$.

Claim. *The graph G^* is chordal.*

In the second case, it is trivial. In the first case, suppose G^* contains a chordless cycle C of length at least 4. Since G is chordal, C contains at least one of u_1, u_2. Suppose first it contains both u_1 and u_2. As they are adjacent in G^*, and contracting them into the vertex u yields an induced subgraph of G, it follows that C has length 4. As in the proof of Lemma 4, it can be seen that the paths corresponding to two consecutive vertices in a C_4 cannot be both vertical. So, suppose that C contains only one of u_1, u_2, say u_1. Since G is chordal, u_2 has to be adjacent to every vertex of $C \setminus N_{G^*}[u_1]$. Since u_1 and u_2 cannot have two non-adjacent common neighbours, at least one of the neighbours of u_1 in C is not adjacent to u_2. Thus, its corresponding path either lies on column y_j having its lower endpoint in row x_t or lies on some row between x_{i+1} and x_t. In either case, this vertex cannot have a common neighbour with u_2, a contradiction. $\diamondsuit$

Now, for every vertex w in $N_{G^*}(v)$, consider the connected component C_w of $G^* - (N_{G^*}[v] - w)$ containing w. Notice that C_w is also chordal contact B_0-VPG and w is non internal in $G^* - (N_{G^*}[v] - w)$. Furthermore, if there are two distinct vertices w and w' in $N_{G^*}(v)$, then C_w and $C_{w'}$ are disjoint. By contradiction, suppose that a vertex x is in the intersection of C_w and $C_{w'}$. Then, there is a path α_1 between w and x, and a path α_2 between x and w'. First, suppose w and w' are non adjacent. Joining both paths we can extract a new induced path α_3 between w and w' which necessarily has length ≥ 3. But then, adding v to α_3 forms an induced cycle with length ≥ 4, a contradiction. On the other hand, if w and w' are adjacent, first remove the edge w and w'. Joining the paths α_1 and α_2 we can extract an induced path α_3 between w and w', which necessarily has length ≥ 4, since G is K_4^--free (see Lemma 5) and, therefore, any vertex adjacent to both w and w' must be also adjacent to v, implying that it does not belong to C_w. Adding the edge between w and w' again, we obtain an induced cycle with length ≥ 4, a contradiction.

Finally, considering the case in which $G^* = G$, it is clear that C_w has at least one vertex less than G, namely v; otherwise, if u was split, the size of G^* is one more than the size of G, but then at least two vertices are removed in C_w, namely v and one between u_1 and u_2 (since there is only one vertex in $N_{G^*}[v]$ that we are not removing).

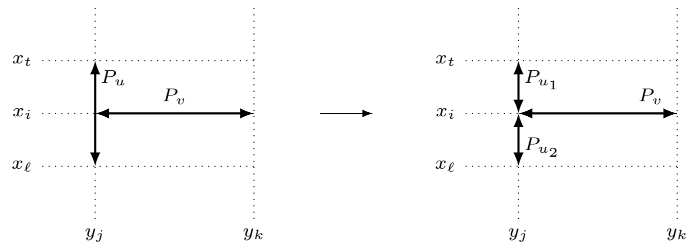


Figure 3: How to split P_u into two paths.

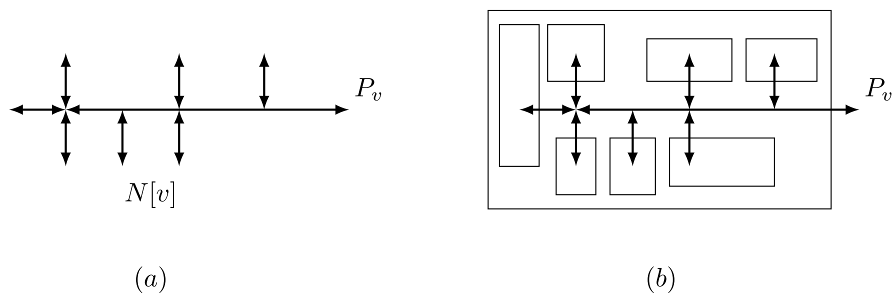


Figure 4: Figure illustrating Lemma 8.

Then, by induction, there exists a contact B_0-VPG representation of C_w, for each such w, with all the paths lying to the left of one free endpoint of P_w. Now, we replace the initial representation of C_w by the new one (the one where all the paths lie to the left of one free endpoint of P_w) by rotating it such that P_w has its free endpoint on the grid-point corresponding to the intersection of P_w and P_v, and belongs to the same side as in the old representation. Notice that we may need to extend the path P_v to the right before doing the replacement of these new representations to assure that they do not overlap. Therefore, by extending if necessary the path P_v a little more to the right, we obtain a contact B_0-VPG representation of G^* in which all the paths lie to the left of one free endpoint of P_v. In case P_u was split into P_{u_1} and P_{u_2}, we now glue these two paths together again. □

Lemma 9. *Let G be a chordal contact B_0-VPG graph. A vertex v in G is internal if and only if in every contact B_0-VPG representation of G, each endpoint of the path P_v either corresponds to the intersection of a representation of K_4 or intersects a path P_w, which represents an internal vertex w, but not at an endpoint of P_w.*

PROOF. The if part is trivial. Assume now that v is an internal vertex of G and consider an arbitrary contact B_0-VPG representation of G. Let P be an

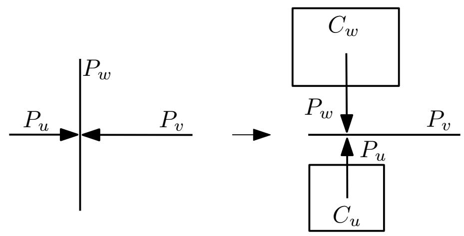


Figure 5: Figure illustrating Lemma 9.

endpoint of the path P_v and K the maximal clique corresponding to all the paths containing the point P. Notice that clearly v is an end in K by definition of K. First, suppose there is a vertex w in K which is not an end. Then, it follows from Remark 3 that the size of K is at most three. Without loss of generality, we may assume that P_v lies on some row and P_w on some column. If w is an internal vertex, we are done. So we may assume now that w is not an internal vertex in G. Consider $G \setminus (K \setminus \{w\})$, and let C_w be the connected component of $G \setminus (K \setminus \{w\})$ containing w. Notice that w is not an internal vertex in C_w either. By Lemma 8, there exists a contact B_0-VPG representation of C_w with all the paths lying to the left of a free endpoint of P_w. Now, replace the old representation of C_w by the new one such that P corresponds to the free endpoint of P_w in the representation of C_w (it might be necessary to refine –by adding rows and/or columns– the grid to ensure that there are no unwanted intersections) and P_w uses the same column as before. Finally, if K had size three, say it contains some vertex u in addition to v and w, then we proceed as follows. Similar to the above, there exists a contact B_0-VPG representation of C_u, the connected component of $G \setminus (K \setminus \{u\})$ containing u, with all the paths lying to the left of a free endpoint of P_u, since u is clearly not internal in C_u. We then replace the old representation of C_u by the new one such that the endpoint of P_u that intersected P_w previously corresponds to the grid-point P and P_u lies on the same column as P_w (again, we may have to refine the grid). This clearly gives us a contact B_0-VPG representation of G. But now we may extend P_v such that it strictly contains the grid-point P and thus, P_v has a free endpoint, a contradiction (see Figure 5). So w must be an internal vertex.

Now, assume that all vertices in K are ends. If $|K| = 4$, we are done. So we may assume that $|K| \leq 3$. Hence, there is at least one grid-edge containing P, which is not used by any paths of the representation. Without loss of generality, we may assume that this grid-edge belongs to some row x_i. If P_v is horizontal, we may extend it such that it strictly contains P. But then v is not internal anymore, a contradiction. If P_v is vertical, then we may extend P_w, where $w \in K$ is such that P_w is a horizontal path. But now we are again in the first case discussed above. □

In other words, Lemma 9 tells us that a vertex v is an internal vertex in a chordal contact B_0-VPG graph if and only if we are in one of the following situations:

- v is the intersection of two cliques of size four (we say that v is of type 1);
- v belongs to exactly one clique of size four and in every contact B_0-VPG representation, v is a middle neighbour of some internal vertex (we say that v is of type 2);
- v does not belong to any clique of size four and in every contact B_0-VPG representation, v is a middle neighbour of two internal vertices (we say that v is of type 3).

Notice that two internal vertices of type 1 cannot be adjacent (except when they belong to a same K_4). Furthermore, an internal vertex of type 1 cannot be the middle-neighbour of some other vertex.

Let $\mathcal{T}$ be the family of graphs containing H_0 (see Figure 2) as well as all graphs that can be partitioned into a nontrivial tree T_0 of maximum degree at most three and the disjoint union of triangles, in such a way that each triangle is complete to a vertex v of T_0 and anticomplete to $T_0 - \{v\}$, every leaf v of T_0 is complete to exactly two triangles, every vertex v of degree two in T_0 is complete to exactly one triangle, and vertices of degree three in T_0 have no neighbours outside T_0 (see Figure 6).

Notice that all graphs in $\mathcal{T}$ are chordal. We denote by $B(T)$ the base tree of T in $\mathcal{T}$.

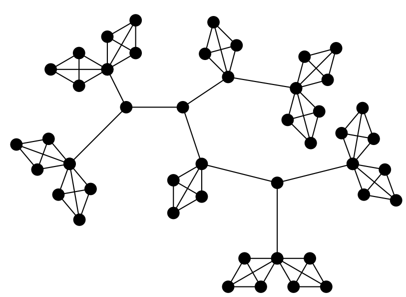


Figure 6: An example of a graph in $\mathcal{T}$.

Lemma 10. *The graphs in $\mathcal{T}$ are not contact B_0-VPG.*

Proof. By Lemma 5, the graph H_0 is not contact B_0-VPG. Consider now a graph $T \in \mathcal{T}$, $T \neq H_0$. Suppose that T is contact B_0-VPG. Consider an arbitrary contact B_0-VPG representation of T. Consider the base tree $B(T)$ and direct an edge uv of it from u to v if the path P_v contains an endpoint of the path P_u (this way some edges might be directed both ways). If a vertex v has degree $d_B(v)$ in $B(T)$, then by definition of the family $\mathcal{T}$, v belongs to $3 - d_B(v)$ K_4's in T. Notice that P_v spends one endpoint in each of these

K_4's. Thus, any vertex v in $B(T)$ has at most $2 - (3 - d_B(v)) = d_B(v) - 1$ outgoing edges. This implies that the sum of out-degrees in $B(T)$ is at most $\sum_{v \in B(T)}(d_B(v) - 1) = n - 2$, where n is the number of vertices in $B(T)$. But this is clearly impossible since there are $n - 1$ edges in $B(T)$ and all edges are directed. □

We will show now how to construct new graphs in $\mathcal{T}$ from others.

Lemma 11. *i) Given $T \in \mathcal{T}$ and $v \in B(T)$ such that v belongs to at least one K_4, say K, then the graph T' constructed by removing the other vertices in K (different from v) and adding one vertex w to $B(T)$, belonging to two copies of K_4 (sharing vertex w), and adjacent to v, belongs to $\mathcal{T}$.*

ii) Given $T_1, T_2 \in \mathcal{T}$, $v_1 \in B(T_1)$ and $v_2 \in B(T_2)$ such that v_1 and v_2 belong to at least one K_4 each, say K_1 and K_2, then the graph T' constructed by removing the other vertices in K_1 and K_2 (different from v_1 and v_2) and adding one vertex w to $B(T_1) \cup B(T_2)$, belonging to a K_4, and adjacent to both v_1 and v_2, belongs to $\mathcal{T}$.

Proof. i) In this case we have $B(T') = B(T) \cup \{w\}$. It is clear that every vertex in $B(T')$ has degree 3 or less, since we only changed the degree of v, which is one less, and the degree of w is one (only adjacent to v in $B(T')$). Moreover, w is a leaf in $B(T')$ and, by construction, it belongs to two copies of K_4 (sharing vertex w). Finally, notice that v has degree 1 or 2 in $B(T)$ since vertices of degree 3 in $B(T)$ does not belong to any K_4. If v is a leaf in $B(T)$, then v is a degree 2 vertex in $B(T')$ and, since we removed the other vertices in K, it belongs to only one K_4 in T'. Otherwise, v has degree 2 in $B(T)$ and therefore, it has degree 3 in $B(T')$ and does not belong to any K_4 in T'. Thus, $T' \in \mathcal{T}$.

ii) In this case we have $B(T') = B(T_1) \cup B(T_2) \cup \{w\}$. The proof follows in the same manner as the previous item. □

For the next lemma we need to consider an orientation of some edges related to a contact B_0-VPG representation of G, given by the following rule. If $v, w \in G$ and v is a middle neighbour of w, then we give the orientation from v to w. Let C_v be the reachable vertices starting from v, including v. Notice that if v is internal, $C_v = \{v\}$ if and only if v is of type 1. Also notice that C_v is independent of the representation for internal vertices. As a consequence of the previous lemma, we can prove the following.

Lemma 12. *Let G be a chordal contact B_0-VPG graph. If a vertex v in G is internal, the graph G' constructed by adding a K_4, say K, containing v to G contains an induced subgraph $T \in \mathcal{T}$. Moreover, $B(T) = C_v$.*

Proof. We will prove this by induction in the number of vertices in C_v. By Lemma 9, v must be of type 1, 2 or 3. As noted before, the base case is when v is of type 1. But then v is the intersection of three cliques of size 4 in G', namely

K and the two cliques in which v is an end; and thus, G' contains $T = H_0$. Therefore $B(T) = \{v\} = C_v$.

Now, if v is of type 2, v is a middle neighbour of exactly one other internal vertex w. Therefore $C_v = C_w \cup \{v\}$. Define G_w as the induced subgraph of the connected component of $G - v$ containing w. Notice that w is still internal in G_w since v is a middle neighbour of w in G. Then, adding a $K' = K_4$ containing w to G_w we obtain a $T_1 \in \mathcal{T}$ induced in G_w (and, therefore, also induced in G) with $B(T_1) = C_w$, by inductive hypothesis applied to w in G_w. By Lemma 11 i), we can construct $T \in \mathcal{T}$ by removing the other vertices in K' (different from w) and adding the vertex v (in G) to $B(T_1)$, which belongs to two copies of K_4 (one is K and the other is the one in which v is an end), and is adjacent to w. Then, T is an induced subgraph of G and we have $B(T) = C_w \cup \{v\} = C_v$. Finally, if v is of type 3, v is a middle neighbour of exactly two other internal vertices w_1 and w_2. The proof continues in the same manner as before, applying inductive hypothesis to the corresponding G_{w_1} and G_{w_2} and then using the second item of Lemma 11. □

Using Lemmas 7–12, we are able to prove the following theorem, which provides a minimal forbidden induced subgraph characterisation of chordal contact B_0-VPG graphs.

Theorem 13. *Let G be a chordal graph. Let $\mathcal{F} = \mathcal{T} \cup \{K_5, K_4^-\}$. Then, G is a contact B_0-VPG graph if and only if G is $\mathcal{F}$-free.*

Proof. Suppose that G is a chordal contact B_0-VPG graph. It follows from Lemma 5 and Lemma 10 that G is $\mathcal{T}$-free and contains neither a K_4^- nor a K_5.

Conversely, suppose now that G is chordal and $\mathcal{F}$-free. By contradiction, suppose that G is not contact B_0-VPG and assume furthermore that G is a minimal non contact B_0-VPG graph. Let v be a simplicial vertex of G (v exists since G is chordal). By Lemma 7, it follows that v has degree three. Consider a contact B_0-VPG representation of $G - v$ and let $K = \{v_1, v_2, v_3\}$ be the set of neighbours of v in G. Since G is K_4^--free, it follows that any two neighbours of v cannot have a common neighbour which is not in K. First suppose that all the vertices in K are ends in the representation of $G - v$. Thus, there exists a grid-edge not used by any path and which has one endpoint corresponding to the intersection of the paths $P_{v_1}, P_{v_2}, P_{v_3}$. But now we may add the path P_v using exactly this grid-edge (we may have to add a row/column to the grid that subdivides this grid-edge and adapt the paths, if the other endpoint of the grid-edge belongs to a path in the representation). Hence, we obtain a contact B_0-VPG representation of G, a contradiction.

Thus, we may assume now that there exists a vertex in K which is not an end, say v_1. Notice that v_1 must be an internal vertex. If not, there is a contact B_0-VPG representation of $G - v$ in which v_1 has a free end. Then, using similar arguments as in the proof of Lemma 9, we may obtain a representation of $G - v$ in which all vertices of K are ends. As described previously, we can add P_v to obtain a contact B_0-VPG representation of G, a contradiction. Now, consider the graph $G - K$. This graph is clearly chordal contact B_0-VPG as being an induced subgraph of $G - v$. Then, by Lemma 12, adding the clique $K \cup \{v\}$ (containing the internal vertex v) to $G - K$ (which gives the graph G) contains an induced subgraph $T \in \mathcal{T}$, a contradiction. □

Interval graphs form a subclass of chordal graphs. They are defined as being chordal graphs not containing any asteroidal triple, i.e. not containing three pairwise non-adjacent vertices such that there exists a path between any two of them avoiding the neighbourhood of the third one. Clearly, any graph in $\mathcal{T}$ for which the base tree has maximum degree three contains an asteroidal triple. On the other hand, H_0 and every graph in $\mathcal{T}$ obtained from a base tree of maximum degree at most two are clearly interval graphs. Denote by $\mathcal{T}'$ the family consisting of H_0 and the graphs of $\mathcal{T}$ whose base tree has maximum degree at most two. We obtain the following corollary which provides a minimal forbidden induced subgraph characterisation of contact B_0-VPG graphs restricted to interval graphs.

Corollary 14. *Let G be an interval graph and $\mathcal{F}' = \mathcal{T}' \cup \{K_5, K_4^-\}$. Then, G is a contact B_0-VPG graph if and only if G is $\mathcal{F}'$-free.*

4. Recognition algorithm

In this section, we will provide a polynomial-time recognition algorithm for chordal contact B_0-VPG graphs which is based on the characterisation given in Section 3. This algorithm takes a chordal graph as input and returns YES if the graph is contact B_0-VPG and, if not, it returns NO as well as a forbidden induced subgraph. The main loop (step 7) will try to find a graph $T \in \mathcal{T}$, $T \neq H_0$. For this purpose, some vertices will be marked and some edges will be directed and coloured. At the beginning all vertices are unmarked and all edges are undirected and uncoloured. We will first give the pseudo-code of our algorithm and then explain the different steps.

Input: a chordal graph $G = (V, E)$;
Output: YES, if G is contact B_0-VPG; NO and a forbidden induced subgraph, if G is not contact B_0-VPG.

1. list all maximal cliques in G;
2. if some edge belongs to two maximal cliques, return NO and K_4^-;
3. if a maximal clique contains at least five vertices, return NO and K_5;
4. label the vertices such that $l(v)$ = number of K_4's that v belongs to;
5. if for some vertex v, $l(v) \geq 3$, return NO and H_0;
6. if $l(v) \leq 1$ $\forall v \in V \setminus \{w\}$ and $l(w) \leq 2$, return YES;
7. while there exists an unmarked vertex v with $2 - l(v)$ outgoing arcs incident to it, do

7.1 mark v as internal;

7.2 direct the edges that are currently undirected, uncoloured, not belonging to a K_4, and incident to v towards v;

7.3 for any two incoming arcs $wv, w'v$ such that $ww' \in E$, colour ww';

8. if there exists some vertex v with more than $2 - l(v)$ outgoing arcs, return NO and find $T \in \mathcal{T}$ by running BFS starting with v, following the outgoing arcs, and adding for each vertex the corresponding K_4's that it belongs to; else return YES.

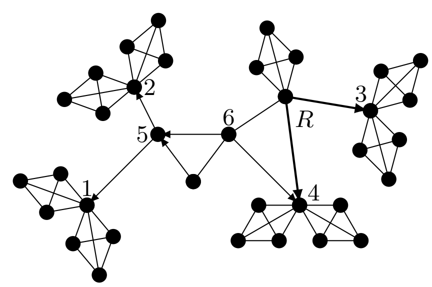


Figure 7: An example of a possible running of the algorithm. The vertices marked in the algorithm are numbered in the order of the marking process. The vertex labeled R corresponds to the root of the tree in the forbidden structure, given in step 8 (whose other vertices are marked as 3 and 4).

Steps 1-5 can clearly be done in polynomial time (see for example [16] for listing all maximal cliques in a chordal graph). Furthermore, it is obvious to see how to find the forbidden induced subgraph in steps 2, 3 and 5. Notice that if the algorithm has not returned NO after step 5, we know that G is $\{K_4^-, K_5, H_0\}$-free. So we are left with checking whether G contains some graph $T \in \mathcal{T}$, $T \neq H_0$. Since each graph $T \in \mathcal{T}$ contains at least two vertices belonging to two K_4's, it follows that if at most one vertex has label 2, G is $\mathcal{T}$-free (step 6), and thus we conclude by Theorem 13 that G is contact B_0-VPG.

During step 7, we detect those vertices in G that, in case G is contact B_0-VPG, must be internal vertices (and mark them as such) and those vertices w that are middle neighbours of internal vertices v (we direct the edges wv from w to v). Furthermore, we colour those edges whose endpoints are middle neighbours of a same internal vertex.

Consider a vertex v with $2 - l(v)$ outgoing arcs. If a vertex v has $l(v) = 2$, then, in case G is contact B_0-VPG, v must be an internal vertex (see Lemma 9). This implies that any neighbour of v, which does not belong to a same K_4 as v, must be a middle neighbour of v. If $l(v) = 1$, this means that v belongs to one K_4 and is a middle neighbour of some internal vertex. Thus, by Lemma 9 we know that v is internal. Similarly, if $l(v) = 0$, this means that v is a middle neighbour of two distinct internal vertices. Again, by Lemma 9 we conclude that v is internal. Clearly, step 7 can be run in polynomial time.

So we are left with step 8, i.e., we need to show that G is contact B_0-VPG if and only if there exists no vertex with more than $2-l(v)$ outgoing arcs. First notice that only vertices marked as internal have incoming arcs. Furthermore, notice that every maximal clique of size three containing an internal vertex has two directed edges of the form wv, $w'v$ and the third edge is coloured, where v is the first of the three vertices that was marked as internal. This is because the graph is K_4^--free and the edges of a K_4 are neither directed nor coloured.

Lemma 15. *Every vertex marked as internal in step 7 has either label 2 or is the root of a directed induced tree (directed from the root to the leaves) where the root w has degree $2 - l(w)$ and every other vertex v has degree $3 - l(v)$ in that tree, namely one incoming arc and $2 - l(v)$ outgoing arcs.*

Proof. By induction in the number of iterations in step 7. In the first iteration, no edge has been directed. Therefore, any vertex marked as internal must have label 2, having zero outgoing edges. Now assume the result is true for any vertex marked before the n-th iteration. Let v be the vertex marked in the n-th iteration. If $l(v) = 2$ we are done. Suppose $l(v) = 1$. Then, there is an outgoing edge from v to a vertex w. Since only vertices marked as internal have incoming arcs, w must be internal. Now, by inductive hypothesis (w was marked in a previous iteration), the result is true for w. If $l(w) = 2$, v is the root of the tree consisting of the two vertices v and w, where v has degree $2 - l(v) = 1$ and w has degree $3 - l(w) = 1$ (one incoming arc). Otherwise, w is the root of a tree T' satisfying the hypothesis of the lemma, but then the tree T constructed from T' by adding v with an outgoing edge to w also clearly satisfies the hypothesis. In a similar manner can be constructed the tree in the case $l(v) = 0$. Finally, let us show that the tree is necessarily induced. Suppose there is an edge not in the tree that joins two vertices of the tree. Since the graph is a block graph, the vertices in the resulting cycle induce a clique, so in particular there is a triangle formed by two edges of the tree and an edge not in the tree. But, as observed above, in every triangle of G having two directed edges, the edges point to the same vertex (and the third edge is coloured, not directed). Since no vertex in the tree has in-degree more than one, this is impossible. □

Based on the lemma, it is clear now that if a vertex has more than $2 - l(v)$ outgoing arcs, then that vertex is the root of a directed induced tree (directed from the root to the leaves), where every vertex v has degree $3 - l(v)$, i.e., a tree that is the base tree $B(T)$ of a graph $T \in \mathcal{T}$. Indeed, notice that every vertex v in a base tree has degree $3 - l(v)$. The fact that tree is induced can be proved the same way as above. This base tree can be found by a breadth-first search from a vertex having out-degree at least $3-l(v)$, using the directed edges. Thanks to the labels, representing the number of K_4's a vertex belongs to, it is then possible to extend the $B(T)$ to an induced subgraph $T \in \mathcal{T}$. This can clearly be implemented to run in polynomial time.

To finish the proof that our algorithm is correct, it remains to show that if G contains an induced subgraph in $\mathcal{T}$, then the algorithm will find a vertex with at least $3 - l(v)$ outgoing arcs. This, along with Theorem 13, says that if the algorithm outputs YES then the graph is contact B_0-VPG (given that the detection of K_5, K_4^- and H_0 is clear). Recall that we know that G is a block graph after step 2. Notice that if a block of size 2 in a graph of $\mathcal{T}$ is replaced by a block of size 4, we obtain either H_0 or a smaller graph in $\mathcal{T}$ as an induced subgraph. Moreover, adding an edge to a graph of $\mathcal{T}$ in such a way that now contains a triangle, then we obtain a smaller induced graph in $\mathcal{T}$. Let G be a block graph with no induced K_5 or H_0. By the remark above, if G contains a graph in $\mathcal{T}$ as induced subgraph, then G contains one, say T, such that no edge of the base tree $B(T)$ is contained in a K_4 in G, and no triangle of G contains two edges of $B(T)$. So, all the edges of $B(T)$ are candidates to be directed or coloured.

In fact, by step 7 of the algorithm, every vertex of $B(T)$ is eventually marked as internal, and every edge incident with it is either directed or coloured, unless the algorithm ends with answer NO before. Notice that by the remark about the maximal cliques of size three and the fact that no triangle of G contains two edges of $B(T)$, if an edge vw of $B(T)$ is coloured, then both v and w have an outgoing arc not belonging to $B(T)$. So, in order to obtain a lower bound on the out-degrees of the vertices of $B(T)$ in G, we can consider only the arcs of $B(T)$ and we can consider the coloured edges as bidirected edges. With an argument similar to the one in the proof of Lemma 10, at least one vertex has out-degree at least $3 - l(v)$.

5. Tree-cographs

In this section, we present a minimal forbidden induced subgraph characterisation for contact B_0-VPG graphs within the class of tree-cographs.

Tree-cographs [24] are a generalisation of cographs, i.e. P_4-free graphs. They are defined recursively as follows: trees are tree-cographs; the disjoint union of tree-cographs is a tree-cograph; and the complement of a tree-cograph is also a tree-cograph.

It follows from the definition that every tree-cograph is either a tree, or the complement of a tree, or the disjoint union of tree-cographs, or the join of tree-cographs. Let us start with the following two trivial facts.

Fact 16. *Every tree is a contact B_0-VPG graph.*

Fact 17. *The disjoint union of contact B_0-VPG graphs is contact B_0-VPG.*

Now let us consider the complement of trees. We obtain the following.

Lemma 18. *Let T be a tree. Then $\overline{T}$ is contact B_0-VPG if and only if it is $\{K_5, K_4^-\}$-free.*

Proof. If $\overline{T}$ is contact B_0-VPG, then it follows from Lemma 5 that $\overline{T}$ is $\{K_5, K_4^-\}$-free.

Suppose now that $\overline{T}$ is $\{K_5, K_4^-\}$-free, then T has stability number at most 4. In particular, it has at most four leaves. Since it does not have co-(K_4-e)'s either, we conclude that T is either a star with at most 4 leaves, a P_4 or a P_5. Hence, $\overline{T}$ is either a $K_4 \cup K_1$, a P_4 or $\overline{P_5}$. Clearly, all these graphs are contact B_0-VPG. □

Using the previous results, we are able to obtain the following characterisation of tree-cographs that are contact B_0-VPG.

Theorem 19. *Let G be a tree-cograph. Then G is contact B_0-VPG if and only if G is $\{K_5, K_{3,3}, H_0, K_4^-\}$-free.*

Proof. If G is contact B_0-VPG, then it follows from Lemma 5 that G is $\{K_5, K_{3,3}, H_0, K_4^-\}$-free.

Suppose now that G is a $\{K_5, K_{3,3}, H_0, K_4^-\}$-free tree cograph on n vertices. We will do a proof by induction on the number of vertices of G. Let us assume the theorem holds for graphs of less than n vertices. If G is a tree, the complement of a tree or the disjoint union of tree-cographs, then the result holds by Facts 16, 17, Lemma 18 and the induction hypothesis. So we may assume now that G is the join of two tree-cographs, say G_1, G_2.

Since G is K_4^--free, both G_1 and G_2 are P_3-free, i.e., they are the disjoint union of cliques. Furthermore, since G is K_5-free, it follows that $\omega(G_1) + \omega(G_2) \leq 4$ and, in particular, none of G_1, G_2 contains a K_4.

First suppose that one of G_1, G_2, say G_1, contains a triangle. Then G_2 contains no K_2. But since G is K_4^--free, G_2 contains no $2K_1$ either. So G_2 is the trivial graph. Now, since G is H_0-free, G_1 contains at most two triangles. But then G is clearly contact B_0-VPG. We show in Figure 8 how to represent the join of the trivial graph and a graph consisting in the disjoint union of at most two triangles, an arbitrary number of edges and isolated vertices as a contact B_0-VPG graph.

Next suppose that $\omega(G_1) = \omega(G_2) = 2$. Since G is K_4^--free, neither G_1 nor G_2 contains $2K_1$. So $G = K_4$, and hence it is contact B_0-VPG.

Suppose now $\omega(G_1) = 2$ and $\omega(G_2) = 1$. Since G is K_4^--free, G_2 contains no $2K_1$, so G_2 is the trivial graph and hence clearly contact B_0-VPG.

Finally, consider the case when $\omega(G_1) = \omega(G_2) = 1$. Since G is $K_{3,3}$-free, it follows that G is either the star $K_{1,n-1}$ or the complete bipartite graph $K_{2,n-2}$. Thus again, G is clearly contact B_0-VPG. □

From the proofs of the previous results, the following fact can be deduced.

Corollary 20. *Every contact B_0-VPG tree-cograph is the disjoint union of trees, $\overline{P_5}$'s, and contact B_0-VPG cographs.*

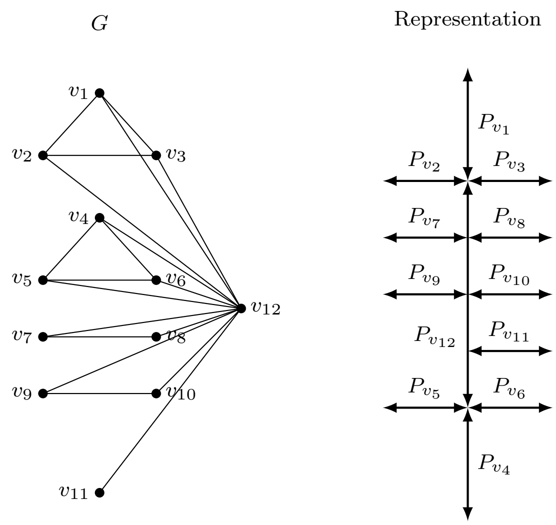


Figure 8: A graph G with G_1 with a most two triangles and $G_2 = K_1$, and a contact B_0-VPG representation of G.

6. P_4-tidy graphs

Let G be a graph and let A be a vertex set that induces a P_4 in G. A vertex v of G is said to be a *partner* of A if $G[A \cup \{v\}]$ contains at least two induced P_4's. The graph G is called P_4*-tidy*, if each vertex set A inducing a P_4 in G has at most one partner [17]. The class of P_4-tidy graphs is an extension of the class of cographs, i.e. P_4-free graphs, and it contains many other graph classes defined by bounding the number of P_4's according to different criteria; e.g., P_4-sparse graphs [21], P_4-lite graphs [22], and P_4-extendible graphs [23].

A *spider* [21] is a graph whose vertex set can be partitioned into three sets S, C, and R, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a clique; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and non-adjacent to all the vertices in S. The triple (S, C, R) is called the *spider partition.* By $\text{thin}_k(H)$ and $\text{thick}_k(H)$ we respectively denote the thin spider and the thick spider with $|C| = |S| = k$ and H the subgraph induced by R. If R is an empty set we denote them by thin_k and thick_k, respectively. Clearly, the complement of a thin spider is a thick spider, and vice versa. A *fat spider* is obtained from a spider by adding a true or false twin of a vertex $v \in S \cup C$. The following theorem characterises P_4-tidy graphs.

Theorem 21. [17] *Let G be a P_4-tidy graph with at least two vertices. Then, exactly one of the following conditions holds:*

1. *G is disconnected.*

2. $\overline{G}$ *is disconnected.*
3. G *is isomorphic to* P_5*,* $\overline{P_5}$*,* C_5*, a spider, or a fat spider.*

This allows us to obtain the following characterisation of contact B_0-VPG P_4-tidy graphs.

Theorem 22. *Let* G *be a* P_4*-tidy graph. Then* G *is contact* B_0*-VPG if and only if* G *is* $\{K_5, K_{3,3}, H_0, K_4^-\}$*-free.*

Proof. If G is a contact B_0-VPG graph, then it follows from Lemma 5 that G is $\{K_5, K_{3,3}, H_0, K_4^-\}$-free.

Suppose that G is a $\{K_5, K_{3,3}, H_0, K_4^-\}$-free P_4-tidy graph on n vertices. We will do a proof by induction on the number of vertices of G. Let us assume the theorem holds for graphs of less than n vertices. It follows from Theorem 21 that G is (i) either disconnected; (ii) or $\overline{G}$ is disconnected; (iii) or G is isomorphic to P_5, $\overline{P_5}$, C_5, a spider, or a fat spider.

If G is disconnected, G is the union of P_4-tidy graphs. Thus the result holds by Fact 17 and the induction hypothesis.

If $\overline{G}$ is disconnected, it follows that G is the join of two P_4-tidy graphs, say G_1, G_2. Then we do exactly the same case analysis as in the proof of Theorem 19.

Now suppose that G is a spider with partition (C, S, R). Since G is K_4^--free, G is necessarily a thin spider. Furthermore, since G is K_5-free, we have $|C| \leq 4$. If $|C| = 4$, then R must be empty. If $|C| = 3$, then $|R| \leq 1$ because G is $\{K_5, K_4^-\}$-free. If $|C| = 2$, then, for the same reasons, $|R| \leq 2$ and if $|R| = 2$, then R induces K_2. Notice that for all these cases, the graph obtained is an induced subgraph of the graph corresponding to the case $|C| = 4$ and $R = \emptyset$. We provide a contact B_0-VPG representation of that case in Figure 9.

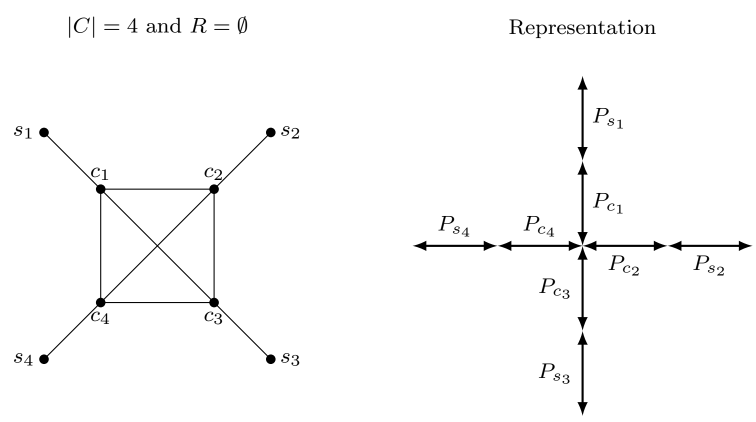


Figure 9: Representation of a thin spider (C, S, R) with $|C| = 4$ and R empty.

Suppose now that G is a fat spider arising from the thin spider with partition (C, S, R). Since G is K_4^--free, it does not arise from adding a true twin to a vertex of C. For the same reason, if $|C| \geq 3$, G does not arise from adding a false twin to a vertex of C, and if $|C| = 2$, we may add a false twin of a vertex of C only if R is empty. We provide a contact B_0-VPG representation for each of these remaining cases in Figure 10.

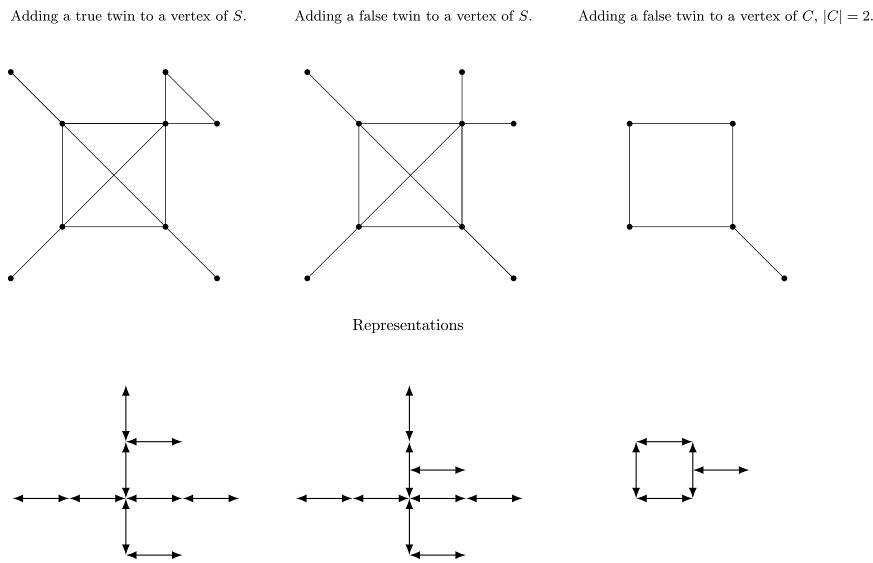


Figure 10: G is a fat spider arising from the thin spider (C, S, R).

Finally, it is easy to see that P_5, $\overline{P_5}$, and C_5 are all contact B_0-VPG graphs. □

For P_4-tidy graphs a linear time recognition algorithm is known [17]. Using the decomposition properties of the class, the characterisation of the possible cases in the proof of Theorem 19 for graphs with disconnected complement, and the possible cases in the proof of Theorem 22 for spiders and fat spiders, we can obtain a linear-time algorithm to determine whether a P_4-tidy graph is contact B_0-VPG. Moreover, we can output a minimal forbidden induced subgraph in the case the answer is no.

7. P_5-free contact B_0-VPG graphs

In this section, we will present a characterisation of P_5-free contact B_0-VPG graphs. Notice that every P_k-free graph, with $1 \leq k \leq 2$, is clearly contact B_0-VPG. Moreover, a P_3-free graph G is a disjoint union of cliques, therefore G is contact B_0-VPG if and only if G is K_5-free.

Concerning P_4-free graphs, we have the following corollary of Theorem 19 or Theorem 22, since P_4-free graphs form a subclass of tree-cographs and P_4-tidy graphs.

Theorem 23. *Let G be a P_4-free graph. Then G is contact B_0-VPG if and only if G is $\{K_5, K_{3,3}, H_0, K_4^-\}$-free.*

Thus, the next graph class to consider is the class of P_5-free graphs. As we will see, the characterisation of P_5-free contact B_0-VPG graphs is much more complex than the characterisation of P_k-free graphs, $k \leq 4$. Consider a P_5-free graph G. If G is chordal, we obtain a characterisation using Theorem 13. Hence, we may assume that G is non chordal. Since G is P_5-free it follows that G contains an induced cycle of length $\ell \in \{4, 5\}$. In what follows, we will first analyse the case when G contains an induced cycle of length four, but no induced cycle of length five.

Lemma 24. *Let G be a non chordal $\{P_5, C_5, K_{3,3}, K_4^-\}$-free graph. Then, there exists an induced cycle C of length four in G such that $N[C] = G$.*

Proof. Since G is not chordal but $\{P_5, C_5\}$-free, it follows that G must contain an induced cycle of length four. Let C_0 be such a cycle induced by the vertices v_1, v_2, v_3, v_4. If $N[C_0] = G$, we are done. Suppose there exists a vertex v at distance two of C_0. So we may assume, without loss of generality, that there is a vertex a adjacent to v_1 and v. It follows from Remark 2 that a must be non-adjacent to at least one of v_2, v_4. Without loss of generality, we may assume that a is non-adjacent to v_4. But then a must be adjacent to v_3, otherwise v, a, v_1, v_4, v_3 induce a P_5, a contradiction. Thus, by Remark 2, a is non-adjacent to v_2.

Now, consider the cycle C_1 induced by the vertices a, v_1, v_2, v_3. If $N[C_1] = G$, we are done. Suppose there is a vertex w at distance two of C_1. Notice that v, a, v_1, v_4 induce a P_4. Thus, w cannot be adjacent to any of v, v_4 otherwise we obtain a P_5 or a C_5, a contradiction. Hence, there exists a vertex $b \neq v, v_4$ adjacent to w and to some vertex in C_1. If b is adjacent to exactly one vertex in C_1 or to exactly two consecutive vertices in C_1, we clearly obtain a P_5, a contradiction. Thus, it follows from Remark 2, that b is adjacent to two nonconsecutive vertices in C_1. We distinguish two cases:

(a) *b is adjacent to a and v_2.* Then b must be adjacent to v_4, otherwise w, b, a, v_1, v_4 induce a P_5, a contradiction. But now v_1, v_3, b, a, v_2, v_4 induce a $K_{3,3}$, a contradiction.

(b) *b is adjacent to v_1 and v_3.* Then b must be adjacent to v, otherwise w, b, v_1, a, v induce a P_5, a contradiction. Now consider the cycle C induced by a, v_1, b, v_3. We claim that $N[C] = G$. Suppose there is a vertex z at distance two of C. Then, following the same reasoning as above, z cannot be adjacent to any of v_4, v, w, v_2, since otherwise we obtain a P_5 or C_5, a contradiction. Thus, as before for vertex b, there exists a vertex c adjacent to z and to two non-adjacent vertices of C. If c is adjacent to v_1 and v_3, then c must also be adjacent to v, otherwise z, c, v_3, a, v induce a P_5. But now v_1, v_3, v, a, b, c induce a $K_{3,3}$, a contradiction. Using the same arguments, we can show that if c is adjacent to a, b, then it must be adjacent to v_2, and again we obtain an induced $K_{3,3}$, a contradiction. Thus z does not exist and hence, $G = N[C]$. □

We will define now the following family of graphs. Start with a cycle C induced by the vertices a_1, b_1, a_2, b_2. Add two (possibly empty) stable sets S_a, S_b, such that every vertex in S_a is adjacent to a_1, a_2 (but not to b_1, b_2), every vertex in S_b is adjacent to b_1, b_2 (but not to a_1, a_2) and S_a is anticomplete to S_b. Furthermore, add two (possibly empty) sets K_a, K_b such that K_a (resp. K_b) is complete to $\{a_1\}$ (resp. $\{b_1\}$) and anticomplete to $\{a_2, b_1, b_2\}$ (resp. $\{a_1, a_2, b_2\}$). Also, every vertex in K_a (resp. K_b) is a simplicial vertex of degree at most three and K_a (resp. K_b) is anticomplete to $S_a \cup S_b \cup K_b$ (resp. $S_a \cup S_b \cup K_a$). Finally, add a (possibly empty) set K_{ab} of vertices forming a clique of size at most two that is complete to $\{a_1, b_1\}$ and anticomplete to the rest of the graph. Moreover, neither of a_1, b_1 can belong to three cliques of size four and only a_1 may belong to two cliques of size four not containing any vertices from K_{ab}. There are no other edges in the graph. Let us denote by $\mathcal{W}_1$ the family of graphs described here before (see Figure 11 for an example).

Let B_1, B_2 and B_3 be the graphs shown in Figure 12. Finally, let $\mathcal{W} = \mathcal{W}_1 \cup \{B_1, B_2, B_3\}$.

Lemma 25. *Let G be a non chordal $\{P_5, C_5, K_5, K_{3,3}, H_0, G_{P_2}, \overline{C_6}, K_4^-\}$-free graph. Then $G \in \mathcal{W}$.*

Proof. Let G be a non chordal $\{P_5, C_5, K_5, K_{3,3}, H_0, G_{P_2}, \overline{C_6}, K_4^-\}$-free graph. It follows from Lemma 24 that there exists an induced cycle C of length four in G such that $N[C] = G$. Let C be induced by vertices a_1, b_1, a_2, b_2. Let S_a (resp. S_b) be the set of vertices adjacent to a_1, a_2 but not b_1, b_2 (resp. to b_1, b_2 but not a_1, a_2). Notice that S_a (resp. S_b) must be a stable set since G is K_4^--free. Furthermore, S_a is anticomplete to S_b. Indeed, if a vertex $v \in S_a$ is adjacent to some vertex $w \in S_b$ then a_1, a_2, w, b_1, b_2, v induce a $K_{3,3}$, a contradiction.

Now, suppose there is a vertex v in G adjacent to only one vertex in C. Without loss of generality, we may assume that v is adjacent to a_1. Then, it is not possible to have a vertex $w \neq v$ in G adjacent only to a_2 in C, since the vertices v, a_1, b_1, a_2, w would induce a P_5 (in case v and w are non-adjacent) or a C_5 (in case v and w are adjacent). Therefore, if there is a vertex $w \neq v$ adjacent to only one vertex in C and different from a_1, then we may assume, without loss of generality, that it is adjacent to b_1. Let K_a (resp. K_b) be the set of vertices adjacent to only a_1 (resp. b_1). If there is a vertex $v \in K_a$ adjacent to a vertex $w \in K_b$, then v, w, b_1, a_2, b_2 induce a P_5, a contradiction. Hence K_a is anticomplete to K_b.

Let us now show that all the vertices in K_a are simplicial. Indeed, suppose that $v \in K_a$ is not simplicial. Then, there exists $w, u \in N(v)$ such that u, w are non-adjacent. It follows from the above that $u, w \in K_a$. But then, v, w, u, a_1 induce a K_4^-, a contradiction. By symmetry, all vertices in K_b are simplicial as well. We will distinguish two cases.

First assume now that G is $\overline{P_5}$-free. Thus every vertex not in C is adjacent to exactly 1 vertex in C, since G is K_4^--free. We claim that S_a is anticomplete to K_a. Indeed, if a vertex $v \in S_a$ is adjacent to some vertex $w \in K_a$, then a_1, b_1, a_2, v, w induce a $\overline{P_5}$, a contradiction. Similarly, S_b is anticomplete to K_b. Next, suppose that some vertex $v \in S_a$ is adjacent to some vertex $w \in K_b$. If S_b is non empty, then for any vertex $u \in S_b$ we obtain a P_5 induced by b_2, u, b_1, w, v, a contradiction. Thus, S_b is empty. Then, we may redefine our cycle C by taking the vertices a_1, b_1, a_2, v. Notice that this cycle also verifies $N[C] = G$. Now, $w \in S_b$ (where S_b is now the set of vertices adjacent to b_1, v but not to a_1, a_2) and $b_2 \in S_a$. We can proceed similarly if S_a is empty and there are adjacent vertices in S_b and K_a. Now, since $S_b \neq \emptyset$, S_a (resp. S_b) is anticomplete to K_b (resp. K_a). Since G is K_5-free, it follows that the degree of the simplicial vertices is at most three. Finally, since G is $\{H_0, G_{P_2}\}$-free, it follows that only a_1 can belong to two cliques of size four and neither of a_1, b_1 can belong to three cliques of size four. Hence, $G \in \mathcal{W}_1$.

Now, suppose that G contains a $\overline{P_5}$ induced by the cycle C and a vertex v adjacent to a_1 and b_1. First, assume there are no other vertices in G adjacent to two consecutive vertices in C. Notice that v cannot be adjacent to any vertex in $S_a \cup S_b \cup K_a \cup K_b$, since G is K_4^--free. Moreover, S_a is anticomplete to K_a. Indeed, if $w \in K_a$ is adjacent to $u \in S_a$, then w, u, a_2, b_1, v induce a P_5, a contradiction. The same applies to K_b and S_b. Finally, we may assume that K_a (resp. K_b) is anticomplete to S_b (resp. S_a) by using the same arguments as above and redefining the cycle C if necessary. Hence, G belongs to $\mathcal{W}_1$.

Next, assume there is another vertex in G (in addition to v) adjacent to two consecutive vertices in C. Notice that a_1 and b_2 (resp. a_2 and b_1) cannot have a common vertex since G is P_5-free. If there is another vertex w adjacent to a_1 and b_1, but there is no vertex adjacent to a_2 and b_2, then w must be adjacent to v, otherwise a_1, b_1, v, w induce a K_4^-, a contradiction. Also, a_1 (resp. b_1) cannot belong to two cliques of size four whose vertices belong to $K_a \cup \{a_1\}$ (resp. $K_b \cup \{b_1\}$), since G is H_0-free. Thus, G belongs to $\mathcal{W}_1$, since G is K_5-free and thus no further vertex is adjacent to both a_1 and b_1. Finally, suppose there is a vertex w adjacent to a_2 and b_2. First notice that v and w are non-adjacent, otherwise v, w, a_1, b_1, a_2, b_2 induce a $\overline{C_6}$, a contradiction. We claim that all the sets S_a, S_b, K_a and K_b must be empty. Indeed, if $u \in S_a$, then u is non-adjacent to w, since G is K_4^--free. But then w, a_2, u, a_1, v induce a P_5, a contradiction. Thus, $S_a = \emptyset$ and by symmetry we also conclude that $S_b = \emptyset$. Now suppose $u \in K_a$. Then the vertices u, a_1, b_1, a_2 and w induce a P_5 (if u, w are non-adjacent) or a C_5 (if u, w are adjacent). Hence $K_a = \emptyset$ and by symmetry $K_b = \emptyset$. If there are no more vertices, G is isomorphic to B_1. If there are more vertices in G, then by using the same arguments as before, these vertices have to be common neighbours of a_1 and b_1, or a_2 and b_2. But then G is necessarily isomorphic to either B_2 or B_3, since G is $\{K_5, \overline{C_6}, K_4^-\}$-free (the same arguments as before apply again).

Finally assume that the $\overline{P_5}$ contained in G is not induced by the cycle C together with some vertex v adjacent to two consecutive vertices in C. The only possibility is that the house is induced by a_1, b_1, a_2, u, w, with $u \in S_a$ and $w \in K_a$ (resp. a_1, b_1, b_2, u, w, with $u \in S_b$ and $w \in K_b$). But then, we may redefine our cycle C by taking the vertices a_1, b_1, a_2, u (resp.a_1, b_1, b_2, u). Clearly this new cycle C also verifies that $N[C] = G$. Thus, we can apply the same arguments as before and show that $G \in \mathcal{W}$. □

Lemma 26. *Every graph in $\mathcal{W}$ is contact B_0-VPG.*

Proof. Let G be a graph in $\mathcal{W}_1$. We construct a contact B_0-VPG representation of G as follows. First represent the main cycle C induced by a_1, b_1, a_2, b_2: P_{a_1} is a horizontal path lying on row x_i; P_{a_2} is a horizontal path lying on row x_j, $j < i$; P_{b_1} is a vertical path lying on column y_k; P_{b_2} is a vertical path lying on column y_ℓ, with $\ell > k + |S_a|$; furthermore, we make sure that b_1 and b_2 are middle-neighbours of a_1 and a_2 is a middle neighbour of b_1 and b_2; finally the paths P_{b_1} and P_{b_2} use column y_k respectively y_ℓ down to row x_t with $t+|S_b| < j$. Now, each vertex in S_a can be represented by a vertical path on some column y_r, with $k < r < \ell$, and every vertex in S_b can be represented by a horizontal path on some row u with $t < u < j$. First assume that $K_{ab} = \emptyset$. Since P_{a_1} has both endpoints free, one can easily represent two cliques of size four, in case a_1 belongs to such cliques and similarly, since P_{b_1} has one endpoint free, one can easily represent one clique of size four, in case b_1 belongs to such a clique. All other vertices in K_a or K_b can clearly be represented by extending enough the paths P_{a_1} and P_{b_1}.

Now, assume that $K_{ab} = \{v\}$. Then, given a contact B_0-VPG representation of $G - v$ as described before, we can easily obtain a contact B_0-VPG representation of G as follows: we add a path P_v lying on column y_k between some row x_q and row x_i, with $i < q$.

Next, assume that $K_{ab} = \{v, v'\}$. Thus, a_1 belongs to at most one clique of size four in $G - \{v, v'\}$ (the vertices of that clique belong to K_a, except for a_1). We obtain a contact B_0-VPG representation as follows. Start with a contact B_0-VPG representation of $G-v'$ as described above. Make sure that all vertices in K_a are represented by paths intersecting P_{a_1} to the right of column y_ℓ (this is clearly always possible, since a_1 belongs to at most one clique of size four whose vertices (except for a_1) belong to K_a). Finally, if necessary, reduce P_{a_1} such that its left endpoint corresponds to the grid point (x_i, y_k) (this is possible since P_{a_1} does not intersect any path to the left of that grid point anymore). Now add P_w as a horizontal path on row x_i with its right endpoint corresponding to the grid point (x_i, y_k).

Finally, if G is one of the graphs B_1, B_2 or B_3, then G is clearly contact B_0-VPG as can be seen in Figure 12(b). Notice that B_1, B_2 are induced subgraphs of B_3. □

From the lemmas above, we conclude the following.

Corollary 27. *Let G be a non chordal $\{P_5, C_5, K_5, K_{3,3}, H, G_{P_2}, \overline{C_6}, K_4^-\}$-free graph. Then G is contact B_0-VPG.*

Let us now focus on P_5-free graphs containing an induced cycle of length five.

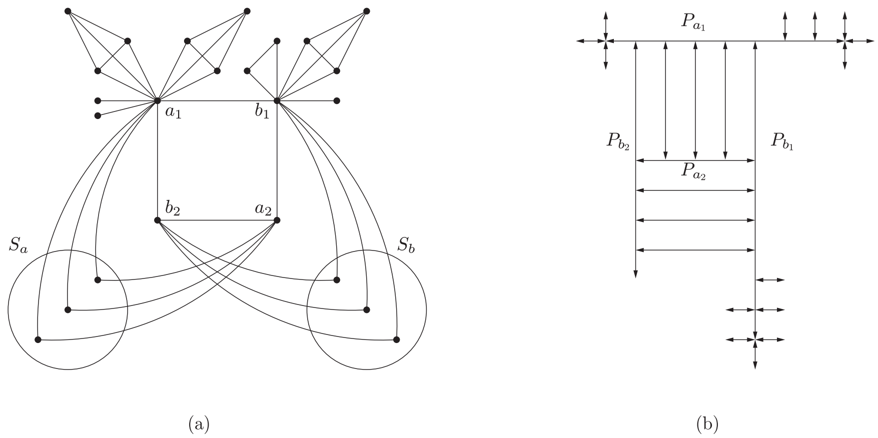


Figure 11: (a) An example of a graph from the family $\mathcal{W}_1$. (b) The corresponding contact B_0-VPG representation.

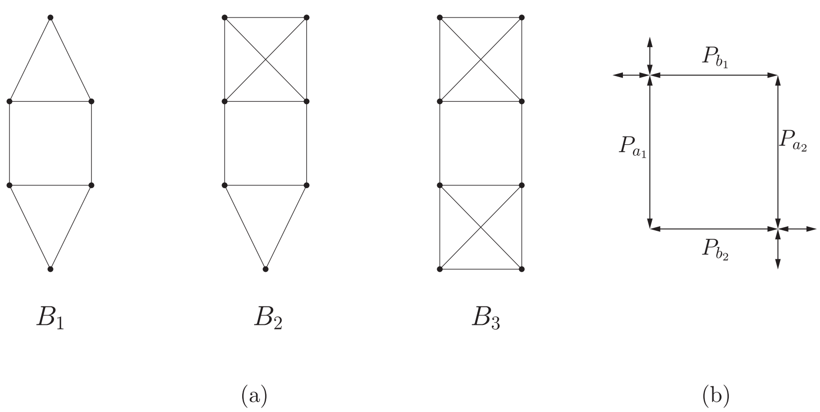
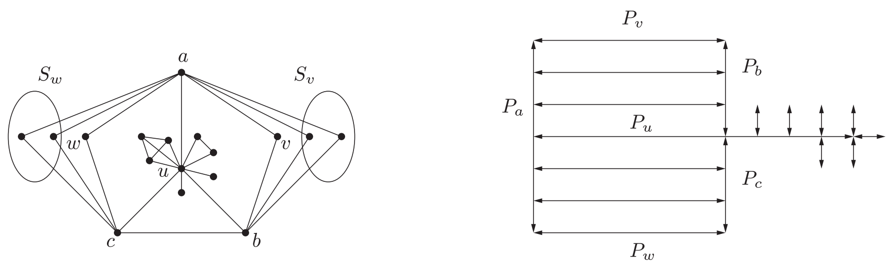


Figure 12: (a) The graphs B_1, B_2 and B_3. (b) A contact B_0-VPG representation of B_3.



(a) (b)

Figure 13: (a) An example of a graph in $\mathcal{L}_2$. (b) The corresponding contact B_0-VPG representation.

Lemma 28. *Let G be a $\{P_5, K_4^-\}$-free graph. Let C be an induced cycle of length five in G such that no vertex is adjacent to exactly three non consecutive vertices in C. Then, $N[C] = G$ and every vertex $v \in N(C)$ is adjacent to exactly two non-consecutive vertices in C.*

Proof. Let C be induced by $v_1, \cdots, v_5$ and let v be a vertex in $N(C)$. It follows from Remark 2 that v cannot be adjacent to three consecutive vertices in C. If v is adjacent to exactly one vertex or to two consecutive vertices in C, then we clearly obtain a P_5, a contradiction. Thus, v has exactly two non consecutive neighbours in C.

Now assume that there exists a vertex u which is at distance two of C. Thus, there is a vertex $w \in N(C)$ adjacent to u and to two non consecutive vertices in C, say v_1, v_3. But then, v, w, v_1, u_5, v_4 induce a P_5, a contradiction. Therefore $N[C] = G$. □

Let $K_{3,3}^*$ be the graph obtained by subdividing exactly one edge in the graph $K_{3,3}$. We will now define several families of graphs. Start with a cycle C of length five induced by the vertices a, v, b, c, w. Add two (possibly empty) stable sets S_v, S_w such that S_v is complete to $\{a, b\}$, S_w is complete to $\{a, c\}$ and S_v is anticomplete to S_w. There are no other edges. Let us denote by $\mathcal{L}_1$ the family of graphs described here before.

Let $G \in \mathcal{L}_1$ and let G' be the graph obtained from G by adding a vertex u adjacent to a, b and c. Furthermore, add a (possible empty) set K_u, such that K_u is complete to $\{u\}$ and anticomplete to $V(C) \cup S_v \cup S_w$. Also, every vertex in K_u is a simplicial vertex of degree at most three. Moreover, u can belong to only one clique of size four. There are no other edges. Let us denote by $\mathcal{L}_2$ the family of graphs described here before (see Figure 13(a) for an example).

Next, consider a graph G' in $\mathcal{L}_2$ with $S_v = S_w = \emptyset$ and u not belonging to any clique of size four. Add a vertex z adjacent to v, w and u. There are no other edges. Let us denote by $\mathcal{L}_3$ the family of graphs obtained that way and let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$.

Finally, let G_1, G_2, G_3 and G_4 be the graphs shown in Figure 14.

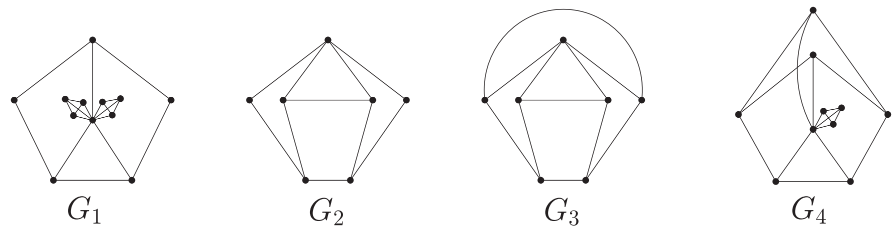


Figure 14: The graphs G_1, G_2, G_3 and G_4.

Lemma 29. *Let G be a $\{P_5, K_5, K^*_{3,3}, \overline{C_6}, G_1, G_2, G_3, G_4, K_4^-\}$-free graph and assume G contains a cycle of length five. Then $G \in \mathcal{L}$.*

Proof. Let C be an induced cycle of length five with vertices a, v, b, c, w. Clearly, no vertex in $N(C)$ is adjacent to exactly one vertex in C or to two consecutive vertices in C, since G is P_5-free. Consider first the vertices adjacent to two non-consecutive vertices in C. For any two vertices u, z that are adjacent to the same two non-consecutive vertices in C, we have that u and z are non-adjacent otherwise we obtain K_4^-, a contradiction. Suppose that there exist vertices u, z such that they have distinct neighbours in C, say u is adjacent to a and b, and z is adjacent to v and c. If u, z are adjacent, then together with the vertices of C, they induce a $K^*_{3,3}$, a contradiction. If u, z are non-adjacent, then u, a, v, z, c induce a P_5, a contradiction. Thus, we may assume now, without loss of generality, that every vertex adjacent to two nonconsecutive vertices in C is either adjacent to both a and b or adjacent to both a an c. Let S_v (resp. S_w) be the set of vertices not in C adjacent to a, b (and not to v, c, w) (resp. a, c (and not to b, v, w)). It follows from the above that S_v and S_w are stable sets. Finally, if there is a vertex $u \in S_v$ adjacent to some vertex $z \in S_w$, then we obtain G_2, a contradiction. So S_v is anticomplete to S_w.

First assume that there exists no vertex in G that is adjacent to three non-consecutive vertices in C. It immediately follows from Lemma 28 that $G = N[C]$ and that every vertex not in C is adjacent to two non-consecutive vertices in C. Thus, $G \in \mathcal{L}_1$.

Now, suppose that there exists a vertex u adjacent to three non-consecutive vertices in C, say a, b and c. We will first show that there cannot be another vertex adjacent to three non-consecutive vertices. If there is another vertex z adjacent to a, b and c, then u and z must be adjacent otherwise u, z, c, b induce a K_4^-, a contradiction. But now the vertices a, u, z and b induce a K_4^-, again a contradiction. Now, suppose z is adjacent to v, b and w. Then, z and u are non-adjacent, since otherwise u, z, b, c induce a K_4^-, a contradiction. But now, the vertices of C together with u and z induce G_3, a contradiction as well. By symmetry, we conclude that z cannot be adjacent to v, w and c. Finally, if z is adjacent to a, v and c, the vertices a, v, z, u, b and c induce a $\overline{C_6}$ if z and u are non-adjacent, a contradiction. But if z and u are adjacent, then u, z, b, c

induce a K_4^-, again a contradiction. By symmetry, z cannot be adjacent to a, w and b. Hence, we conclude that u is the unique vertex adjacent to three non-consecutive vertices in C.

Now we will distinguish several cases, depending on which vertices u is adjacent to. First, assume that u is adjacent to a, b and c, and that $S_v \cup S_w$ is non empty. Notice that u cannot be adjacent to any vertex in $S_v \cup S_w$, since G is K_4^--free. It follows from Remark 2 and the fact that G is P_5-free that any vertex in G not belonging to $V(C) \cup S_v \cup S_w \cup \{u\}$ has to be adjacent to u and anticomplete to $V(C) \cup S_v \cup S_w$. Let $K_u = N(u) \setminus V(C)$ be the set of these vertices and consider $z \in K_u$. Then z is simplicial. Indeed, if z is not simplicial, it follows that there exist vertices $z', z'' \in K_u \cap N(z)$ such that z', z'' are non-adjacent. But then z, z', z'', u induce K_4^-, a contradiction. Furthermore, since G is K_5-free, it follows that every vertex $z \in K_u$ has degree at most three. Finally, notice that u can only belong to at most one clique of size four, since G is G_1-free. Thus, we conclude that $G \in \mathcal{L}_2$.

Notice that if $S_v = S_w = \emptyset$, we can relabel the vertices in C such that u is adjacent to a, b and c, and we obtain a graph in $\mathcal{L}_2$ as before. Thus, we may assume, without loss of generality, that there is a vertex $z \in S_v$. Now, we will consider different cases:

- If u is adjacent to v, b and w, or if u is adjacent to a, v and c, then we obtain G_2 (notice that z and u cannot be adjacent since the graph is K_4^--free),a contradiction.

- If u is adjacent to a, b and w, then $S_w = \emptyset$, otherwise a, v, b, c, u, w, t, where $t \in S_w$, induce G_2 a contradiction. Now, we can relabel the vertices in C such that u is adjacent to a, b and c, without changing S_v, and we obtain a graph in $\mathcal{L}_2$ as before.

- If u is adjacent to v, c and w, and z is non-adjacent to u, then z, a, v, u, c induce a P_5, a contradiction. So z and u must be adjacent. Notice again that $S_w = \emptyset$. Indeed, if $t \in S_w$, then t, a, v, b, u, c, w induce G_2, a contradiction. Moreover, $|S_v| = 1$: if $z' \in S_v$, $z \neq z'$, then z' must be adjacent to u as well, but now v, z, z', a, b, u induce a $K_{3,3}$, a contradiction. So we can relabel the vertices in C such that u is adjacent to a, b, c. With this new labeling, $S_v = S_w = \emptyset$ and z is adjacent to v, w and u. Clearly, any vertex not belonging to $V(C) \cup \{u, z\}$ has to be adjacent to u, since G is P_5-free. Let K_u be the set of these vertices. Using the same arguments than above, one can show that ever vertex in K_u is simplicial and have degree at most three since the graph is K_5-free. Finally, u cannot belong to a clique of size four, since G is G_4-free. So we conclude that $G \in \mathcal{L}_3$.□

Lemma 30. *Every graph in $\mathcal{L}$ is contact B_0-VPG.*

Proof. Let $G \in \mathcal{L}_1$. We construct a contact B_0-VPG representation of G as follows. Vertex b is represented by a path P_b lying on column y_j between rows x_k and x_t, with $t > k + |S_v|$; vertex c is represented by a path P_c lying on

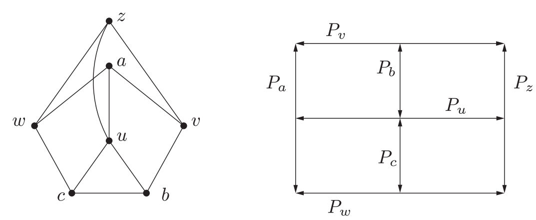


Figure 15: A graph in $\mathcal{L}_3$ and the corresponding contact B_0-VPG representation.

column y_j between rows x_t and x_ℓ, with $\ell > t + |S_w|$; vertex a is represented by a path P_a lying on column y_i, $i < j$, between rows x_k and x_ℓ; vertex v is represented by a path P_v lying on row x_k between rows y_i and y_j and vertex w is represented by a path P_w lying on row x_ℓ between rows y_i and y_j. Now each vertex in S_v is represented by a path between columns y_i and y_j lying on one of the $|S_v|$ rows between x_k and x_t, and each vertex in S_w is represented by a path between columns y_i and y_j lying on one of the $|S_w|$ rows between x_t and x_ℓ.

If $G \in \mathcal{L}_2$, consider a representation of $G - (K_u \cup \{u\})$ as described above. Now, it is possible to add P_u on row x_t, such that b and c are middle-neighbours of u, and u is a middle-neighbour of a. If u belongs to one clique of size four, then it is possible to represent this clique using the right endpoint of P_u. All the other vertices of K_u can easily be represented by eventually extending the path P_u to the right.

Finally, if $G \in \mathcal{L}_3$, consider the contact B_0-VPG representation of the graph shown in Figure 15. Clearly, it is possible to add the paths representing the vertices of K_u, since u does not belong to any clique of size four. $\square$

Lemma 31. *The graphs $K^*_{3,3}, \overline{C_6}, G_1, G_2, G_3, G_4$ are not contact B_0-VPG.*

Proof. Consider the graph $K_{3,3}$ with vertices a, c, e on one side of the bipartition and b, d, f on the other side. Assume that the edge ef is subdivided to obtain $K^*_{3,3}$. Consider the cycle induced by the vertices a, b, c, d. Following the same approach as in Lemma 5, we may assume that P_a, P_c are horizontal paths, P_b, P_d are vertical paths and P_e is a horizontal path lying inside the rectangle, and P_f is a vertical path lying outside the rectangle. But now it is clearly impossible to add a path intersecting P_e and P_f without intersecting any other path. Thus, $K^*_{3,3}$ is not B_0-VPG.

Next consider the graph $\overline{C_6}$ with vertex set a, b, c, d, v, w such that a, b, c, d induce a cycle of length four, v is a common vertex of a and b, w is a common neighbour of c and d, and v is adjacent to w. If $\overline{C_6}$ is contact B_0-VPG, then we may assume that in a contact B_0-VPG representation, the paths P_a, P_c are horizontal and the paths P_b, P_d are vertical. Since b, c, v, w induce a cycle of length four, we conclude from the above that P_v has to be horizontal. But since

a, d, v, w induce a cycle of length four as well, we also conclude that P_v has to be vertical, a contradiction. Hence, $\overline{C_6}$ is not B_0-VPG.

Suppose now that the graph G_1 is contact B_0-VPG. Without loss of generality, we may assume that P_u lies on some row x_i. Since u belongs to two cliques of size four, it follows from Remark 3 that both endpoints of P_u are not free. Thus, a, b and c are middle neighbours of u, i.e. P_a, P_b, P_c are necessarily vertical paths. Thus, P_v, P_w must be horizontal paths, but this is impossible since no two paths can cross. We conclude that G_1 is not contact B_0-VPG.

Using similar arguments, we conclude that if G_4 is contact B_0-VPG, then b, c have to be middle neighbours of u, u has to be a middle neighbour of a and P_v, P_w have to be horizontal paths. But now it is clearly impossible to add P_z such that it intersects P_v, P_w, P_u without crossing any path. Hence, G_4 is not contact B_0-VPG.

Finally, consider the graphs G_2, G_3 and suppose that they are contact B_0-VPG. First consider a contact B_0-VPG representation of $G_2 - v$ (resp. $G_3 - v$). Since t is adjacent to three non-consecutive vertices of a induced cycle of length five, we may assume, without loss of generality, that we have the following configuration: P_a, P_c, P_z are horizontal paths with P_a, P_z lying on a same row; P_b, P_w are vertical paths; P_t is a vertical path with one endpoint corresponding to the endpoints of P_a, P_z that intersect; t is a middle neighbour of c. But now it is clearly impossible to add a path representing vertex v, since it has to intersect P_a and P_b. Therefore, G_2, G_3 are not contact B_0-VPG. □

We are now ready to prove the main result of this section.

Theorem 32. *Let G be a P_5-free graph. Let $\mathcal{G} = \{K_5, H_0, G_{P_2}, K_{3,3}, K^*_{3,3}, \overline{C_6}, G_1, G_2, G_3, G_4, K_4^-\}$. Then G is contact B_0-VPG if and only if G is $\mathcal{G}$-free.*

Proof. For the only if part, we use Theorem 13, Lemma 5 and Lemma 31.

Suppose now that G is a P_5-free graph which is also $\mathcal{G}$-free. If G is chordal, the result follows from Theorem 13, since G is $\mathcal{F}$-free (indeed, the graphs in $\mathcal{F}$ different from H_0 and G_{P_2} contain an induced P_5). Now, assume that G is not chordal. If G is C_5-free, by Corollary 27, G is contact B_0-VPG. Similarly, if G contains a C_5, by Lemmas 29 and 30, G is also contact B_0-VPG. □

8. Conclusions and Future work

In this paper, we considered some special graph classes, namely chordal graphs, tree-cographs, P_4-tidy graphs and P_5-free graphs. We gave a characterisation by minimal forbidden induced subgraphs of those graphs from these families that are contact B_0-VPG. Moreover, we presented a polynomial-time algorithm for recognising chordal contact B_0-VPG graphs based on our characterisation. Notice that for the other graph classes considered here, the characterisation immediately yields a polynomial-time recognition algorithm.

In order to get a better understanding of the structure of general contact B_0-VPG graphs, one way could be to find further characterisations by forbidden induced subgraphs of contact B_0-VPG graphs within other interesting classes. Since classical graph problems are difficult in contact B_0-VPG graphs (see for instance [13]), these further insights in their structure may lead to good approximation algorithms for these problems.

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