Sobre relaciones de incertidumbre generalizadas

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II Jornadas de Fundamentos de Cuántica: Aspectos Epistemológicos

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Principio de Heisenberg

Imposibilidad de que determinados pares de magnitudes físicas sean conocidos con precisión arbitraria

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Magnitudes conjugadas (complementarias):

- $X \sim \Psi(x), \quad f_X(x) = |\Psi(x)|^2$ (densidad de probabilidad)
- $P \sim \widehat{\Psi}(p) = (2\pi)^{-\frac{d}{2}} \int \Psi(x) e^{-\imath p^t x} dx, \quad f_P(p) = |\widehat{\Psi}(p)|^2 \qquad (\hbar = 1)$

$$\Delta X . \Delta P \ge \frac{1}{2}, \quad E\left[\|X\|^2\right] E\left[\|P\|^2\right] \ge \frac{1}{4}, \quad 4E\left[X X^t\right] - E\left[P P^t\right]^{-1} \ge 0$$

Igualdad (solo) en el caso Gaussiana.

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Procesado de señales:

- representación espectral: $s(t) = \int S(\nu) e^{2\imath\pi\nu t} dt$ (energía norm.)
- dispersión $\Delta s = \sqrt{\left\langle \left(s - \langle s \rangle_t\right)^2 \right\rangle_t}$,

  $\Delta s . \Delta S \geq \frac{1}{4\pi}$ representación tiempo-frecuencia óptima...

Igualdad (solo) en el caso Gaussiana.

Caso continuo vs discreto y mixto

Caso continuo

$$\widehat{\Psi}(p) = (2\pi)^{-\frac{d}{2}} \int \Psi(x) e^{-\imath p^t x} dx \, : \, \frac{1}{2} \overset{\text{ipp}}{=} \left| \langle x \, , \, \Psi \nabla \widehat{\Psi}^* \rangle \right| = \left| \langle x \Psi^*, \nabla \widehat{\Psi}^* \rangle \right| \overset{\text{CS}}{\leq} \Delta X . \Delta P$$

CS: igualdad sii $\nabla \Psi \propto x \Psi^* \ldots$

CASO CONTINUO

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CASO DISCRETO (n ESTADOS, $(x,p) \in \{1, \ldots, n\}^2$)

$$\widehat{\Psi}(p) = n^{-\frac{1}{2}} \sum_x \Psi(x) \, e^{-\frac{2\imath\pi p x}{n}} \, : \quad \begin{array}{l} \Psi(x) = \delta_{k,x} \\ \Psi(x) = \frac{1}{\sqrt{n}} \end{array} \Rightarrow \Delta X . \Delta P = 0$$

Caso continuo vs discreto y mixto

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Caso mixto ($p \in [0\,;\,2\pi)$)

$$\widehat{\Psi}(p) = (2\pi)^{-\frac{1}{2}} \sum_x \Psi(x)\, e^{-\imath p x} : \quad \begin{array}{l} \Psi(x) = \delta_{k,x} \\ \left(\Psi(x) = \frac{1}{\sqrt{n}}\right) \end{array} \Rightarrow \Delta X.\Delta P = 0$$

Formulaciones cuantitativas

TIPO	CONTINUO (SAT.)	DISCRETO (SAT.)	MIXTO (SAT.)
$\frac{E\left[\lVert X\rVert^2\right]}{d}\,\frac{E\left[\lVert P\rVert^2\right]}{d} \geq \mathcal{H}_g$	$\mathcal{H}_g = \frac{1}{4}$ $(\mathcal{N})$	$\emptyset$	$\emptyset$

Otras formulaciones: por qué?

- Existen variables con varianza infinita (ej. α-stable $f \propto \frac{1}{1+x^2}$)
- Casos discreto y mixto: no existe una cota no trivial
- Varias medidas de dispersión, información o complejidad

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- Entropías,
- Información de Fisher,
- Momentos estadísticos de cualquier orden
- ...

Entropía de Shannon

$$H = -\sum\!\!\!\!\!\!\int_{\Omega} p \log p$$

Termodinámica (Boltzmann)

Comunicación (Shannon)

MaxEnt: $|\Omega|$ finito $\rightsquigarrow p$ unif.

MaxEnt dado $\sigma^2 \rightsquigarrow p \propto \mathrm{e}^{-\frac{x^2}{2\sigma^2}}$

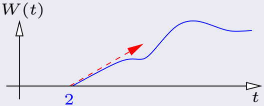
Potencia entrópica $N = \frac{1}{2\pi e} \exp\left(\frac{2}{d} H\right)$

Caso continuo

Beckner'75: $\|\widehat{\Psi}\|_t \le C_{s,t}^d \|\Psi\|_s$ con $C_{s,t} = \left(\frac{2\pi}{t}\right)^{\frac{1}{2t}} \left(\frac{2\pi}{s}\right)^{-\frac{1}{2s}}$, $\frac{1}{t} + \frac{1}{s} = 1$, $t \ge 2$

Bialynicki-Birula & Mycielski '75: $W(t) = C_{s,t}^d \|\Psi\|_s - \|\widehat{\Psi}\|_t$

$$\left.\begin{array}{ll} W(t) \ge 0 & \text{(Beckner)} \\ W(2) = 0 & \text{(Parseval)} \end{array}\right\} \Rightarrow \left.\frac{dW}{dt}\right|_{t=2} \ge 0$$



Da $\frac{H(X)+H(P)}{d} \ge 1 + \log \pi$ i.e. $N(X)N(P) \ge \frac{1}{4}$

Formulación entrópica

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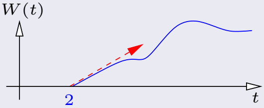

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Casos discreto & mixto

Young-Hausdorff: $\|\widehat{\Psi}\|_t \leq C^{\frac{1}{2t} - \frac{1}{2s}} \|\Psi\|_s$ con $C = n \quad (d)$, $\quad C = 2\pi \quad (m)$

Da $N(X)N(P) \geq \frac{n^2}{4\pi^2 e^2} \quad (d)$ & $N(X)N(P) \geq \frac{1}{e^2} \quad (m)$

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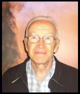
Bialynicki-Birula vs Kennard



Maximizando la entropía condicionada por la "varianza" $E[\|X\|^2]$:

- $N(X) \leq N(G_X)$ donde $G_X \sim \mathcal{N}\left(\cdot\,,\ \frac{E[\|X\|^2]}{d} I\right)$
- $N\left(\mathcal{N}(\cdot\,,\ R)\right) = |R|^{\frac{1}{d}}$

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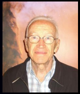

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Entonces: $$\frac{E[\|X\|^2]}{d}\,\frac{E[\|P\|^2]}{d} = N(G_X)N(G_P) \ge N(X)N(P) \ge \frac{1}{4}$$

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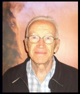

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Formulación de Bialynicki-Birula más fuerte que la de Kennard

Nota: G_X & G_P son conjugadas.

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$N(X)\,N(P) \ge \mathcal{B}_m$	$\mathcal{B}_m = \frac{1}{4}$	$(\mathcal{N})$	$\mathcal{B}_m = \frac{n^2}{4\pi^2 e^2}$	$(\mathcal{U})$	$\mathcal{B}_m = \frac{1}{e^2}$	$(\mathcal{U})$

Entropía de Shannon vs de Rényi

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MaxEnt dado $\sigma^2 \rightsquigarrow p \propto \mathrm{e}^{-\frac{x^2}{2\sigma^2}}$

Entropía de Rényi

$$H_\lambda = \frac{1}{1-\lambda} \log \sum\!\!\!\!\!\!\int_\Omega p^\lambda, \quad \lambda \ge 0$$

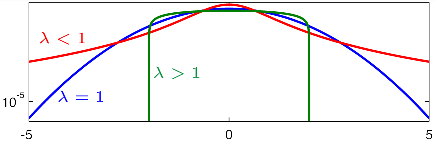
Promed. (Shannon) $\rightsquigarrow$ à la Nagumo

Sis. no equil., multifrac., proc. señales...

MaxEnt: $|\Omega|$ finito $\rightsquigarrow p$ unif.

λ tiene un sentido de zoom

MaxEnt dado $\sigma^2 \rightsquigarrow p \propto \left(1 + \frac{(1-\lambda)\, x^2}{(3\lambda-1)\sigma^2}\right)_+^{\frac{1}{\lambda-1}}$



Potencia de entropía $N_\lambda = \frac{1}{2\pi e} \exp\left(\frac{2}{d} H_\lambda\right)$

Una formulación à la Rényi

- Beckner/Young-Hausdorff: $\|\widehat{\Psi}\|_t \le C^d \, \|\Psi\|_s$
- $\dfrac{1}{2\pi e} \, \|\Psi\|_s^{\frac{4s}{d(2-s)}} = N_{\frac{s}{2}}(X) \qquad \& \qquad \dfrac{1}{2\pi e} \, \|\widehat{\Psi}\|_t^{\frac{4t}{d(2-t)}} = N_{\frac{t}{2}}(P)$
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Potencia $\frac{s}{2-s}$ en ambos lados de la desigualdad & $\alpha = s/2$ da

$$N_\alpha(X) N_{\alpha^\star}(P) \ge \frac{C^{\frac{2\alpha}{\alpha-1} - \frac{2\alpha^\star}{\alpha^\star-1}}}{4\pi^2 e^2} \quad \text{con} \quad \alpha^\star = \frac{\alpha}{2\alpha-1}, \quad \alpha \ge \frac{1}{2}$$

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Nota: $\alpha \to 1 \;\Rightarrow\; \alpha^\star \to 1$

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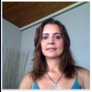
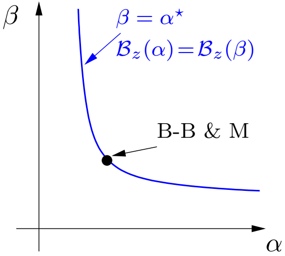
$N_\alpha(X)N_\beta(P), \quad \beta \neq \alpha^\star$?





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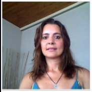

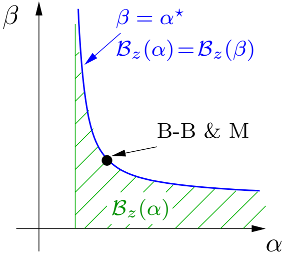

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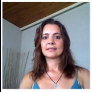

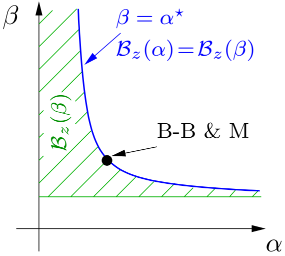

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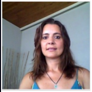

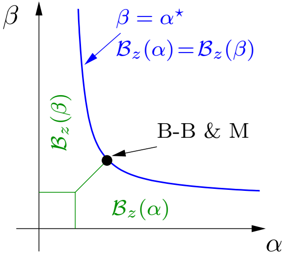

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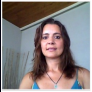
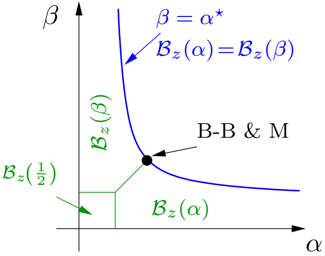




$f(\lambda) = N_\lambda$ decrece con λ:

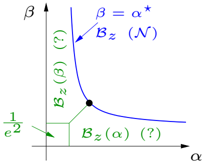
- $N_\alpha(X) N_\beta(P) \geq \mathcal{B}_z(\alpha), \quad \alpha \geq \frac{1}{2}$
- $N_\alpha(X) N_\beta(P) \geq \mathcal{B}_z(\beta), \quad \beta \geq \frac{1}{2}$
- $\mathcal{B}_z(\alpha)$ crece con α:
  $N_\alpha(X) N_\beta(P) \geq \mathcal{B}_z(\max(\alpha, \beta)), \ (\alpha, \beta) \notin \left[0\,;\, \frac{1}{2}\right]^2$
- $N_\alpha(X) N_\beta(P) \geq \mathcal{B}_z\left(\frac{1}{2}\right), \ (\alpha, \beta) \in \left[0\,;\, \frac{1}{2}\right]^2$

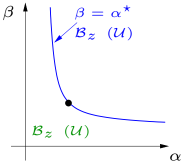
$$N_\alpha(X) N_\beta(P) \geq \mathcal{B}_z\left(\max\left(\alpha, \beta, \frac{1}{2}\right)\right), \quad 0 \leq \beta \leq \alpha^\star, \quad 0 \leq \alpha \leq \beta^\star$$

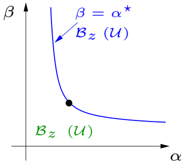
FORMULACIONES CUANTITATIVAS


Scanned at the American Institute of Physics

TIPO	CONTINUO (SAT.)	DISCRETO (SAT.)	MIXTO (SAT.)
$\frac{E\left[\Vert X\Vert^2\right]}{d}\,\frac{E\left[\Vert P\Vert^2\right]}{d} \ge \mathcal{H}_g$	$\mathcal{H}_g = \frac{1}{4}$ $(\mathcal{N})$	$\emptyset$	$\emptyset$
$\Uparrow$ MaxEnt cond. a var.			
$N(X)\,N(P) \ge \mathcal{B}_m$	$\mathcal{B}_m = \frac{1}{4}$ $(\mathcal{N})$	$\mathcal{B}_m = \frac{n^2}{4\pi^2 e^2}$ $(\mathcal{U})$	$\mathcal{B}_m = \frac{1}{e^2}$ $(\mathcal{U})$
$\Uparrow \alpha \to 1$			
$N_\alpha(X)\,N_{\alpha^\star}(P) \ge \mathcal{B}_z$	$\mathcal{B}_z = \frac{\alpha^{\frac{1}{\alpha-1}}\,\alpha^{\star\,\frac{1}{\alpha^\star-1}}}{4e^2}$ $(\mathcal{N})$	$\mathcal{B}_z = \frac{n^2}{4\pi^2 e^2}$ $(\mathcal{U})$	$\mathcal{B}_z = \frac{1}{e^2}$ $(\mathcal{U})$
$N_\alpha(X)\,N_\beta(P) \ge \mathcal{Z}_p$	$\mathcal{Z}_p = \mathcal{B}_z(\max(\alpha,\beta,\frac{1}{2}))$ $(?)$	$\mathcal{Z}_p = \mathcal{B}_z$ $(\mathcal{U}$	$\mathcal{Z}_p = \mathcal{B}_z$ $(\mathcal{U}$







Qué pasa si $\beta > \alpha^\star$?

Qué pasa si $\beta > \alpha^\star$?

Caso mixto

??

Qué pasa si $\beta > \alpha^\star$?

Caso mixto

??

Caso discreto

- Landau-Pollack: $\arccos\sqrt{\max_x |\Psi(x)|^2} + \arccos\sqrt{\max_p |\widehat{\Psi}(p)|^2} \geq \arccos\left(\frac{1}{\sqrt{n}}\right)$
- Mínimo de $N_\infty(X)N_\infty(P) = \dfrac{1}{4\pi^2 e^2 (\max_x |\Psi(x)|^2 \max_p |\widehat{\Psi}(p)|^2)^2}$ cond. a L-P

$$N_\alpha(X)N_\beta(P) \geq N_\infty(X)N_\infty(P) \geq \frac{4n^2}{\pi^2 e^2 (1+\sqrt{n})^4}$$

Qué pasa si $\beta > \alpha^\star$?

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$$N_\alpha(X)N_\beta(P) \geq N_\infty(X)N_\infty(P) \geq \frac{4n^2}{\pi^2 e^2 (1+\sqrt{n})^4}$$

Caso continuo

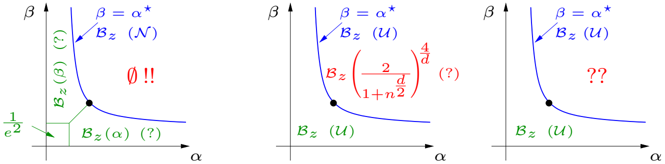
Contraejemplo: $\Psi(x) = \pi^{-\frac{d}{4}} \left(\Gamma\left(\frac{d+\nu}{2}\right)\right)^{\frac{1}{2}} \left(\Gamma\left(\frac{\nu}{2}\right)\right)^{-\frac{1}{2}} \left(1 + \|x\|^2\right)^{-\frac{d+\nu}{4}}$, $\nu > 0$

- $\beta > \max(\alpha^\star, 1)$, $\nu \to \frac{d(\beta-1)}{\beta} \Rightarrow \begin{cases} N_\alpha(X) < +\infty \\ N_\beta(P) \to 0 \end{cases} \Rightarrow N_\alpha(X)N_\beta(P) \to 0$
- $1 > \beta > \alpha^\star$, $\nu \to 0 \Rightarrow N_\alpha(X)N_\beta(P) \propto \left(\Gamma\left(\frac{\nu}{2}\right)\right)^{\frac{\alpha}{\alpha-1}+\frac{\beta}{\beta-1}} \to 0$

$N_\alpha(X)N_\beta(P)$ puede ser arbitrariamente pequeño

Formulaciones cuantitativas

TIPO	CONTINUO (SAT.)	DISCRETO (SAT.)	MIXTO (SAT.)
$\frac{E\left[\Vert X\Vert^2\right]}{d}\,\frac{E\left[\Vert P\Vert^2\right]}{d} \ge \mathcal{H}_g$	$\mathcal{H}_g = \frac{1}{4}$ $(\mathcal{N})$	$\emptyset$	$\emptyset$
$\Uparrow$ MaxEnt cond. a var.			
$N(X)\,N(P) \ge \mathcal{B}_m$	$\mathcal{B}_m = \frac{1}{4}$ $(\mathcal{N})$	$\mathcal{B}_m = \frac{n^2}{4\pi^2 e^2}$ $(\mathcal{U})$	$\mathcal{B}_m = \frac{1}{e^2}$ $(\mathcal{U})$
$\Uparrow \alpha \to 1$			
$N_\alpha(X)\,N_{\alpha^\star}(P) \ge \mathcal{B}_z$	$\mathcal{B}_z = \frac{\alpha^{\frac{1}{\alpha-1}}\,{\alpha^\star}^{\frac{1}{\alpha^\star-1}}}{4e^2}$ $(\mathcal{N})$	$\mathcal{B}_z = \frac{n^2}{4\pi^2 e^2}$ $(\mathcal{U})$	$\mathcal{B}_z = \frac{1}{e^2}$ $(\mathcal{U})$
$N_\alpha(X)\,N_\beta(P) \ge \mathcal{Z}_p$	$\mathcal{Z}_p = \mathcal{B}_z(\max(\alpha, \beta, \frac{1}{2})), \emptyset$ (?)	$\mathcal{Z}_p = \mathcal{B}_z \times \cdots$ $(\mathcal{U}, ?)$	$\mathcal{Z}_p = \mathcal{B}_z, ?$ $(\mathcal{U}, ?)$



Regreso a los momentos. . .

Caso continuo

- $\max N_\lambda(Z)$ dado $E[\|Z\|^l]$: $\qquad \dfrac{E\left[\|Z\|^l\right]^{\frac{2}{l}}}{d} \geq N_\lambda(Z)\mathcal{M}(l,\lambda)$

CASO CONTINUO

- $\max N_\lambda(Z)$ dado $E[\|Z\|^l]$: $\quad \dfrac{E\left[\|Z\|^l\right]^{\frac{2}{l}}}{d} \geq N_\lambda(Z)\mathcal{M}(l,\lambda)$
- $\forall (a,b) \in \mathbb{R}_+^2, \;\; \alpha > \frac{d}{a+d}, \beta > \frac{d}{b+d},$

$$\frac{E[\|X\|^a]^{\frac{2}{a}}}{d} \frac{E\left[\|P\|^b\right]^{\frac{2}{b}}}{d} \geq N_\alpha(X)\mathcal{M}(a,\alpha)N_\beta(P)\mathcal{M}(b,\beta) \geq \mathcal{Z}_p(\alpha,\beta)\mathcal{M}(a,\alpha)\mathcal{M}(b,\beta)$$

Caso continuo

- $\max N_\lambda(Z)$ dado $E[\|Z\|^l]$: $$\frac{E\left[\|Z\|^l\right]^{\frac{2}{l}}}{d} \ge N_\lambda(Z)\mathcal{M}(l,\lambda)$$
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- J.-C. Angulo et al.: $\mathcal{A}(a,b) = \mathcal{Z}_p(1,1)\mathcal{M}(a,1)\mathcal{M}(b,1)$ (a partir de BBM)

Regreso a los momentos...

Caso continuo

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- Si $a \geq b$, cota máx. para $\beta = \alpha^\star, \alpha \in D = \left(\max\left(\frac{1}{2}, \frac{d}{d+\max(a,b)}\right)\;;\;1\right]$ & sim,

$$\frac{E\left[\|X\|^a\right]^{\frac{2}{a}}}{d}\,\frac{E\left[\|P\|^b\right]^{\frac{2}{b}}}{d} \geq \mathcal{Z}_m = \max_{\alpha\in D} \mathcal{B}_z(\alpha)\mathcal{M}(\max(a,b),\alpha)\mathcal{M}(\min(a,b),\alpha^\star)$$

Regreso a los momentos...

Caso continuo

- $\max N_\lambda(Z)$ dado $E[\|Z\|^l]$: $\quad \dfrac{E\left[\|Z\|^l\right]^{\frac{2}{l}}}{d} \geq N_\lambda(Z)\mathcal{M}(l,\lambda)$
- $\forall (a,b) \in \mathbb{R}_+^2, \quad \alpha > \frac{d}{a+d}, \beta > \frac{d}{b+d},$

$$\frac{E[\|X\|^a]^{\frac{2}{a}}}{d}\frac{E\left[\|P\|^b\right]^{\frac{2}{b}}}{d} \geq N_\alpha(X)\mathcal{M}(a,\alpha)N_\beta(P)\mathcal{M}(b,\beta) \geq \mathcal{Z}_p(\alpha,\beta)\mathcal{M}(a,\alpha)\mathcal{M}(b,\beta)$$

- J.-C. Angulo et al.: $\mathcal{A}(a,b) = \mathcal{Z}_p(1,1)\mathcal{M}(a,1)\mathcal{M}(b,1)$ (a partir de BBM)
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$$\frac{E\left[\|X\|^a\right]^{\frac{2}{a}}}{d}\frac{E\left[\|P\|^b\right]^{\frac{2}{b}}}{d} \geq \mathcal{Z}_m = \max_{\alpha\in D} \mathcal{B}_z(\alpha)\mathcal{M}(\max(a,b),\alpha)\mathcal{M}(\min(a,b),\alpha^\star)$$

Casos discreto y mixto

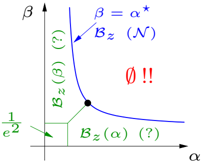
$\Psi(x) = \delta_{k,x}$

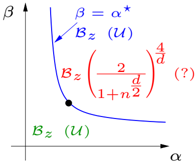
$(\Psi(x) = \frac{1}{\sqrt{n}})$

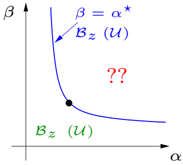
$$\Rightarrow \frac{E\left[\|X\|^a\right]^{\frac{2}{a}}}{d}\frac{E\left[\|P\|^b\right]^{\frac{2}{b}}}{d} = 0$$

Formulaciones cuantitativas

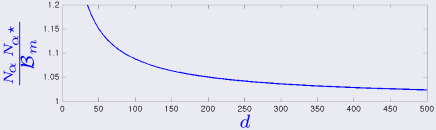
Tipo	Continuo (sat.)	Discreto (sat.)	Mixto (sat.)
$\frac{E\left[\|X\|^2\right]}{d}\,\frac{E\left[\|P\|^2\right]}{d} \geq \mathcal{H}_g$	$\mathcal{H}_g = \frac{1}{4}$ $(\mathcal{N})$	$\emptyset$	$\emptyset$
$\Uparrow$ MaxEnt cond. a var.			
$N(X)\,N(P) \geq \mathcal{B}_m$	$\mathcal{B}_m = \frac{1}{4}$ $(\mathcal{N})$	$\mathcal{B}_m = \frac{n^2}{4\pi^2 e^2}$ $(\mathcal{U})$	$\mathcal{B}_m = \frac{1}{e^2}$ $(\mathcal{U})$
$\Uparrow \alpha \to 1$			
$N_\alpha(X)\,N_{\alpha^\star}(P) \geq \mathcal{B}_z$	$\mathcal{B}_z = \frac{\alpha^{\frac{1}{\alpha-1}}\,{\alpha^\star}^{\frac{1}{\alpha^\star-1}}}{4e^2}$ $(\mathcal{N})$	$\mathcal{B}_z = \frac{n^2}{4\pi^2 e^2}$ $(\mathcal{U})$	$\mathcal{B}_z = \frac{1}{e^2}$ $(\mathcal{U})$
$N_\alpha(X)\,N_\beta(P) \geq \mathcal{Z}_p$	$\mathcal{Z}_p = \mathcal{B}_z(\max(\alpha,\beta,\frac{1}{2}))$, $\emptyset$ (?)	$\mathcal{Z}_p = \mathcal{B}_z \times \cdots$ $(\mathcal{U},?)$	$\mathcal{Z}_p = \mathcal{B}_z$, ? $(\mathcal{U},?)$
$\Downarrow$ MaxEnt cond. a $\mathcal{M}$			
$\frac{E[\|X\|^a]^{\frac{2}{a}}}{d}\,\frac{E\left[\|P\|^b\right]^{\frac{2}{b}}}{d} \geq \mathcal{Z}_m$	$\mathcal{Z}_m$ numerica (??)	$\emptyset$	$\emptyset$





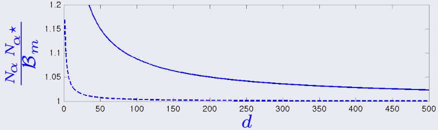


BBM: saturación asintótica



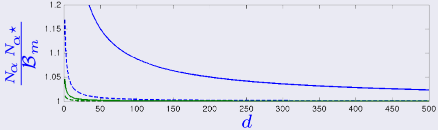
Cauchy , $\alpha = 1$

BBM: saturación asintótica



Cauchy & Laplaciana, $\alpha = 1$

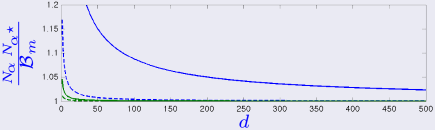
BBM: saturación asintótica



Cauchy & Laplaciana, $\alpha = 1$

Cauchy & Laplaciana, $\alpha = 3$

BBM: saturación asintótica



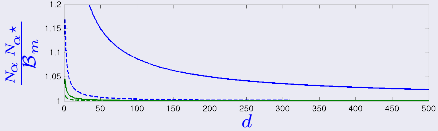
Cauchy & Laplaciana, $\alpha = 1$

Cauchy & Laplaciana, $\alpha = 3$

Student$-t$: resultado analítico (argumentos tipo convexidad)

ALGUNOS RESULTADOS

BBM: SATURACIÓN ASINTÓTICA



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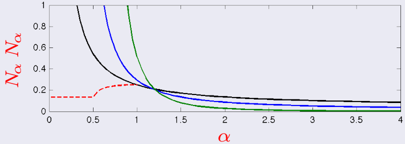
CON INDICES NO CONJUGADOS

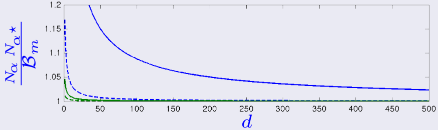


Student$-t$ con ν grados de libertad, $\nu = 1$, $\nu = 3$ & $\nu \to \infty$ (Gauss).

$$\nu \to \frac{d(\alpha-1)}{\alpha} \Rightarrow N_\alpha(X) N_\alpha(P) \to 0$$

Algunos resultados

BBM: saturación asintótica



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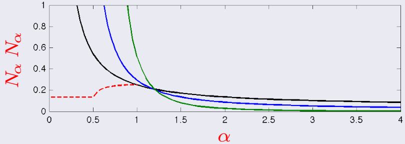

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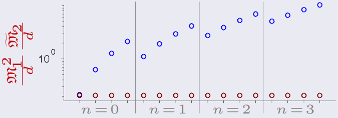
Momentos: oscilador armónico cuántico

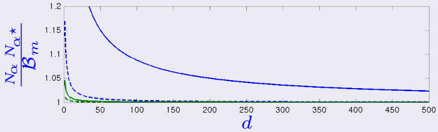


Schrödinger: $\left[-\frac{1}{2}\nabla_x^2 + V(r)\right]\Psi = E_n\Psi$

Oscilador: $V(r) = \frac{1}{2}r^2$

Algunos resultados

BBM: saturación asintótica



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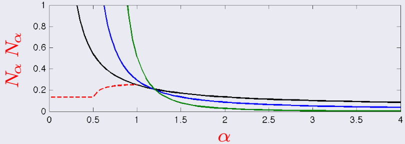

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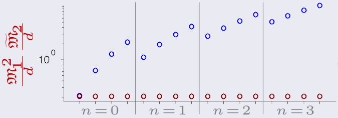
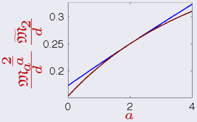


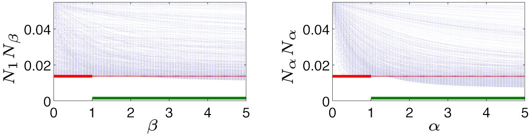


Schrödinger: $\left[-\frac{1}{2}\nabla_x^2 + V(r)\right]\Psi = E_n\Psi$

Oscilador: $V(r) = \frac{1}{2}r^2$

Generalizando más. . .

- Cotas óptimas? máx. ?



Generalizando más. . .

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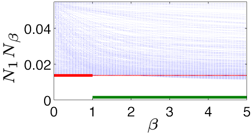
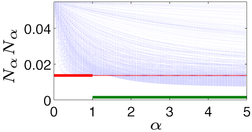
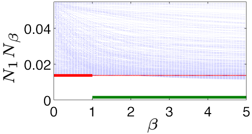


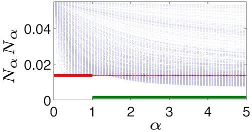


- Obs. no conjugados: $\widetilde{\psi} = T\psi$, T unit., solapamiento $c = \max_{i,j} |T_{i,j}|$

GENERALIZANDO MÁS...

- Cotas óptimas? máx. ?





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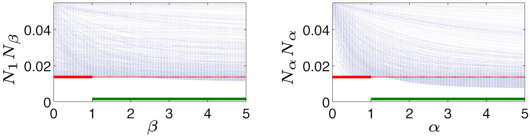
  - Maassen-Uffink'88: $N(X)\, N(\widetilde{X}) \geq \frac{1}{4\pi^2 e^2 c^4}$

    (mejorado por DeVicente et al.'08)

Generalizando más. . .

- Cotas óptimas? máx. ?



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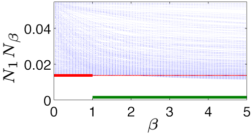
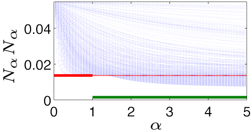
  - Maassen-Uffink'88: $N(X)\,N(\widetilde{X}) \geq \dfrac{1}{4\pi^2 e^2 c^4}$

    (mejorado por DeVicente et al.'08)
  - Bosyk & al'12: qubit, $N_2(X)\,N_2(\widetilde{X}) \geq \dfrac{4}{\pi^2 e^2 (1+c^2)^4}$

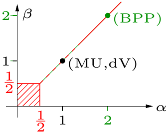
Generalizando más...

- Cotas óptimas? máx. ?





- Obs. no conjugados: $\widetilde{\psi} = T\psi$, T unit., solapamiento $c = \max_{i,j} |T_{i,j}|$
  - Maassen-Uffink'88: $N(X)\, N(\widetilde{X}) \geq \dfrac{1}{4\pi^2 e^2 c^4}$ (mejorado por DeVicente et al.'08)
  - Bosyk & al'12: qubit, $N_2(X)\, N_2(\widetilde{X}) \geq \dfrac{4}{\pi^2 e^2 (1+c^2)^4}$
  - Caso del qubit, $(\alpha, \beta) \in \mathbb{R}_+^2$ (Bosyk et al. arxiv'12, (α, α) & $c = 1/\sqrt{2}$)



Generalizando más. . .

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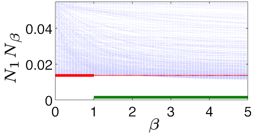

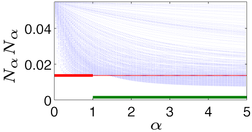

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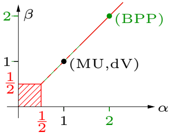
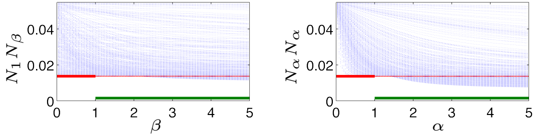


  - $(\alpha, \beta) \in \mathbb{R}_+^2$, $n > 2$?

Generalizando más...

- Cotas óptimas? máx. ?



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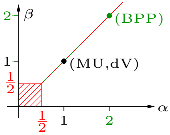

  - $(\alpha, \beta) \in \mathbb{R}_+^2$, $n > 2$?
- Caso continuo $\widetilde{\Psi} = \mathcal{T}\Psi$? Momentos?

¡Gracias!

Gràcies!

Thank you!

DANKE!

Merci!

你很