On the irregular part of $V-$statistics multifractal spectra for systems with non-uniform specification

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Abstract

Let (X, f) be a dynamical system with X a compact metric space. Let X^r be the product of $r-$copies of X, $r \geq 1$, and $\Phi : X^r \to \mathbf{R}$. The multifractal decomposition for $V-$statistics for Φ, f is defined as

$E_{\Phi,f}(\alpha) = \left\{ x : \lim_{n\to\infty} \frac{1}{n^r} \sum_{0\leq i_1,\ldots,i_r \leq n-1} \Phi\left(f^{i_1}(x), \ldots, f^{i_r}(x)\right) = \alpha \right\}$. The set of points $x \in X$, for which the limit does not exist is called *the irregular part, or historic set*, of the spectrum.

In this article we analyze the irregular part of the $V-$statistics for systems satisfying a weak form of the known Bowen specification property, called the *non-uniform specification property.* This concept was introduced by P. Varandas and allows to work in a nonuniformly hyperbolic context.

KEYWORDS: non-uniform specification, $V-$statistics, multifractal spectra

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I.

II. INTRODUCTION

The multiple ergodic averages appeared as a dynamical version of the Szemeredi theorem in combinatorial number theory. This analogy was pointed out by Furstenberg[8] who studied ergodic averages in a measure-preserving probability space $(X, \mathcal{B}, \mu, f)$ of the form

$$\frac{1}{N-M} \sum_{n=M}^{N-1} \mu\left(A \cap f^n A \cap ... \cap f^{kn} A\right), \tag{1}$$

where $A \in \mathcal{B}$ and $j \in \mathbf{N}$. Furstenberg established that if $\mu(A) > 0$ then $\liminf_{N\to\infty} \dfrac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap f^n A \cap ... \cap f^{jn} A) > 0$. This relevant result serves to prove by arguments from Ergodic Theory the Szemeredi theorem, which states that if S is a set of integers with positive upper density then S contains arithmetic progressions of arbitrary length.

The multifractal analysis of Vstatistics treated by Fan, Schmeling and Wu[5] was motivated by the problems on convergence of multiple ergodic averages. Let us consider a topological dynamical system (X, f), with X a compact metric space and f a continuous map. Let $X^r = X \times ... \times X$ be the product of $r-$copies of X with $r \geq 1$, if $\Phi : X^r \to \mathbf{R}$ is a continuous map, then let

$$V_\Phi(n, x) = \frac{1}{n^r} \sum_{1 \leq i_1, ..., i_r \leq n} \Phi\left(f^{i_1}(x), ..., f^{i_r}(x)\right). \tag{2}$$

These averages are called the $V-$*statistics of order* r *with kernel* Φ. For the idea of Vstatistics from a Statistical point of view and its relationship with the Ustatistics see section 2 of[5].

Ergodic limits of the form

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Phi\left(f^{i_1}(x), ..., f^{i_r}(x)\right),$$

were studied among others by Furstenberg[8], Bergelson[1] and Bourgain[2].

The multifractal decomposition for the spectra of $V-$statistics is

$$E_{\Phi}(\alpha) = \left\{ x : \lim_{n\to\infty} V_{\Phi}(n,x) = \alpha \right\}.$$

Fan, Schemeling and Wu[5] have obtained the following variational principle for dynamical systems with the specification property.:

$$h_{top}(E_{\Phi}(\alpha)) = \sup \left\{ h_{\mu}(f) : \int \Phi d\mu^{\otimes r} = \alpha \right\}, \tag{3}$$

where h_{top} is the topological entropy for non-compacts nor invariant sets and $h_{\mu}(f)$ is the measure-theoretic entropy of μ. Here $\mu^{\otimes r}$ means $\mu \times ... \times \mu$, $r-$times. This generalizes the variational principle established by Takens and Verbitski for $r = 1$[11].

The *irregular part* of the spectrum, or *historic set*, is the set of points x for which $\lim_{n\to\infty} V_{\Phi}(n,x)$ does not exist. We denote this set by E_{Φ}^{∞}, so that the space X can be decomposed as

$$X = \bigcup_{\alpha\in\mathbf{R}} E_{\Phi}(\alpha) \cup E_{\Phi}^{\infty}.$$

In a recent paper [9]we have studied the irregular part of the multifractal decomposition of $V-$statistics, in order to determine its dimension. We proved that for topological dynamical systems with the property of specification, if the irregular part of the spectrum of multiple ergodic averages, or $V-$statistics is non-empty then it has the same topological entropy as the whole space X.

The objective of the present article is to extend the above result for systems satisfying a weak form of the specification property, known as *non-uniform specification* condition. This notion was introduced by P. Varandas[14] and is satisfied, for instance, by non-uniformly quadratic maps and for the so called Viana maps, which are a robust class of multidimensional non-uniformly hyperbolic functions[14]. The result to be proved is

Theorem 1.1: Let (X, f) be a dynamical system with the property of non-uniform specification. Let $\Phi \in C(X^r)$, $r \geq 1$, and let $E_{\Phi}^{\infty}(\alpha)$ be the irregular part of the spectrum of multiple ergodic averages $V_{\Phi}(n,x)$. Then $E_{\Phi}(\alpha)$ is empty or $h_{top}(E_{\Phi}^{\infty}(\alpha)) = h_{top}(X)$.

For the proof of the result in [9], we used the variational principle for systems with the specification property of Fan, Schemeling and Wu. The key point for the demonstration of this variational principle is the saturadness of. This means that

$$h_{top}\left(G\left(\mu\right)\right)=h_{\mu}\left(f\right), \qquad (4)$$

where by $G(\mu)$ is denoted the set of $\mu-$generic points. Bowen [3] proved the inequality

$$h_{top}\left(G\left(\mu\right)\right)\leq h_{\mu}\left(f\right),$$

while in [7] was proved the opposite inequality, i.e. the saturadness of dynamical systems with specification. Thus to extend our result of [9] to systems with the non-uniform specification property we must prove that such systems are saturated. Once proved this, following [7], we obtain

Theorem 1.2: Let (X,f) be a dynamical system with the property of non-uniform specification. Let $\Phi\in C\left(X^r\right)$, $r\geq 1$, then

$$h_{top}(E_{\Phi}\left(\alpha\right))=\sup\left\{h_{\mu}\left(f\right):\int\Phi d\mu^{\otimes r}=\alpha\right\}.$$

With the theorem 1.2 and the saturadness, following similar lines than in [9] we obtain theorem 1.1.

Remark: The case $r=1$ was proved in [15]. In that article the proof is not based on the saturadness, unlike herein.

III. PRELIMINARIES

Firstly let us recall the Bowen definition of topological entropy of sets: Let $f:X\rightarrow X$, with X a compact metric space, for $n\geq 1$ the dynamical metric, or Bowen metric, is $d_n\left(x,y\right)=\max\left\{d\left(f^i\left(x\right),f^i\left(y\right)\right):i=0,1,...,n-1\right\}$. We denote by $B_{n,\varepsilon}\left(x\right)$ the ball of centre x and radius ε in the metric d_n. Let $Z\subset X$ and let $\mathcal{C}\left(n,\varepsilon,Z\right)$ be the collection of finite or countable coverings of the set Z by balls $B_{m,\varepsilon}\left(x\right)$ with $m\geq n$. Let

$$M\left(Z,s,n,\varepsilon\right)=\inf_{\mathcal{B}\in\mathcal{C}(n,\varepsilon,Z)}\sum_{B_{m,\varepsilon}(x)\in\mathcal{B}}\exp\left(-sm\right),$$

and set

$$M\left(Z,s,\varepsilon\right)=\lim_{n\rightarrow\infty}M\left(Z,s,n,\varepsilon\right).$$

There is an unique number $\overline{s}$ such that $M(Z,s,\varepsilon)$ jumps from $+\infty$ to 0. Let

$$H(Z,\varepsilon) = \overline{s} = \sup\{s : M(Z,s,\varepsilon) = +\infty\} = \inf\{s : M(Z,s,\varepsilon) = 0\},$$

and

$$h_{top}(Z) = \lim_{\varepsilon\to 0} H(Z,\varepsilon). \tag{5}$$

The number $h_{top}(Z)$ is the *topological entropy* of Z.

A dynamical system (X,f) has the *non-uniform specification property* if the following condition holds, for $\delta > 0$, $0 < \varepsilon < \delta$, $n \in \mathbf{N}$, $x \in X$, there exists an integer $M(x,n,\varepsilon)$ such that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup \frac{1}{n} M(x,n,\varepsilon) = 0,$$

and such that is verified, given $x_1, x_2, ..., x_k \in X$, $n_1, n_2, ..., n_k \in \mathbf{N}$, if $M_i \geq M(x_i, n_i, \varepsilon)$ then there is a point $z \in X$ such that

$$d_{n_1}(x_1, z) < \varepsilon$$

and

$$d_{n_i}\left(f^{\sum_{j=1}^{i-1}(n_j+M_j)}(z), x_i\right) < \varepsilon.$$

By $\mathcal{M}(X)$ we denote the space of measures in X, and by $\mathcal{M}_{inv}(X,f)$ the space of $f-$invariant measures on X. The space $\mathcal{M}(X)$ can be endowed with a metric D compatible with the metric in X, in the sense that $D(\delta_x, \delta_y) = d(x,y)$, where δ is the point mass measure. More precisely the metric considered in $\mathcal{M}(X)$ will be

$$D(\mu,\nu) = \sum_{n=1}^{\infty} \frac{\left|\int \varphi_n d\mu - \int \varphi_n d\nu\right|}{2^n \|\varphi_n\|_\infty},$$

where $\{\varphi_n\}$ is a dense set in $C(X)$. We denote by $B_R(\mu)$ the ball of center μ and radius R in the above metric. The topology induced by this metric is the weak $*-$ topology, and if X is compact then $\mathcal{M}(X)$ is compact in the weak topology. The weak convergence is the convergence in the metric which induces the weak topology.

The so called empirical measures on X associated to the dynamical system (X,f) are

$$\mathcal{E}_n(x) = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}.$$

Here δ is the point mass measure. We denote the weak limits of the sequence$\{\mathcal{E}_n(x)\}$ by $V(x)$. Since X is compact, $V(x) \neq \varnothing$. If μ is a measure on X then a point $x \in X$ is $\mu-$generic if $V(x) = \{\mu\}$, by $G(\mu)$ is denoted the set of $\mu-$generic points.

Following [7] the set of generic points can be characterized in the following way. Let $\{p_i\}$ be a sequence of numbers with $\sum\limits_{i=1}^{\infty} p_i = 1$ and let $\{s_i\}$ be a sequence in ℓ^∞. The sequence $\{s_i = s_{n,i}\}_i$ converges to $\alpha = (\alpha_i) \in \ell^\infty$ in the weak $*-$ topology if and only if $\lim_{n\to\infty} |s_{n,i} - \alpha_i| = 0$. Let $\{\varphi_1, \varphi_2, ...\}$ a dense subset in unit ball of $C/X)$, for a fixed $\mu \in \mathcal{M}_{inv}(X, f)$, let $\alpha = (\alpha_1, \alpha_2, ...)$, with $\alpha_i = \int \varphi_i d\mu$- Thus

$$G(\mu) = \left\{x : \lim_{n\to\infty}\sum_{i=1}^{\infty} p_i \left|S_n(\varphi_i(x)) - \alpha_i\ \right| = 0\right\}$$

Lemma 2.1([16],[14]): For any $\mu \in \mathcal{M}_{inv}(X, f)$, $0 < \delta < 1$, $0 < \gamma < 1$, there is a measure ν such that $\nu = \sum\limits_{j=1}^{k} \lambda_i \nu_i$, where each ν_j is ergodic and $\sum\limits_{j=1}^{k} \lambda_j = 1$, and such that

i) $h_\nu(f) \geq h_\mu(f) - \gamma$.

ii) $\sum\limits_{i=1}^{\infty} p_i \left|\int \varphi_i d\mu - \int \varphi_i d\nu\right| < \delta$, where $\{\varphi_i\}$ and $\{p_i\}$ are sequences like above.

Let $N \geq 1$ and

$$Y_j(N) = \left\{x : \sum_{i=1}^{\infty} p_i \left|S_n(\varphi_i(x)) - \int \varphi_i d\nu_j\right| < \delta, \text{ for } n > N\right\},$$

where $S_n(\varphi_i(x)) = \sum\limits_{k=0}^{n-1} \varphi_i\left(f^k(x)\right)$. By the Birkhoff ergodic theorem we have that

$$\lim_{n\to\infty}\sum_{i=1}^{\infty} p_i \left|S_n(\varphi_i(x)) - \int \varphi_i d\nu_k\right| = 0,\ \nu_k - a.e.,$$

and for sufficiently large N holds $\nu_j(Y_j(N)) > 1 - \gamma$.

Let $\alpha = (\alpha_1, \alpha_2, ...) \in \ell^\infty$ and $\Theta = \{\varphi_1, \varphi_2, ...\}$ be a dense subset in unit ball of $C(X)$. Set

$$\Lambda_\Theta(\alpha) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\lim_{n\to\infty}\sup\frac{1}{n}\log N(\alpha, \delta, \epsilon, n), \tag{6}$$

where $N(\alpha, \delta, \epsilon, n)$ is the minimal number balls $B_{n,\varepsilon}(x)$ needed to cover the set

$$X_\Theta(\alpha, \delta, n) = \left\{x : \sum_{i=1}^{\infty} p_i \left|S_n(\varphi_i(x)) -\ \alpha_i\right| < \delta,\ \alpha = (\alpha_i) \in \ell^\infty\right\}$$

IV. CONSTRUCTION OF A FRACTAL SET AND PROOF OF SATURADNESS

The next step is the construction of a fractal set F, such that $F \subset G(\mu)$. For the construction is followed [7], [11],[**?**] or [4], Let $\{n_k\}$ be a sequence of positive integers and $\{N_k\}$ be an increasing sequence of integers with $N_k \to \infty$ and

$$N_1 = 1,\ N_2 \geq 2^{n_1 + \max_{x\in S_1} M(x,n_1,\varepsilon/4) + n_3 + \max_{x\in S_3} M(x,n_3,\varepsilon/16)}.$$

Let $\{S_k\}$ be a sequence of finite subsets of X, and $\{n_k\}$ be a sequence of positive integers. Let $\varepsilon > 0$ and assume that $d_{n_k}(x,y) > 5\varepsilon$, for any$x \neq y \in S_k$. Sequences of sets $\{D_k\}$ and $\{L_k\}$ are constructed in the following way: Set $D_1 = S_1$, let $x_1, ..., x_{N_k} \in S_k$, $\varepsilon > 0$, by the non-uniform specification property, there exists a $y = y(x_1, ..., x_{N_k})$ such that

$$d_{n_k}\left(x_j, f^{a_j}(y)\right) < \varepsilon/2^k,$$

with

$$a_j = (j-1)\left(n_k + \max_{x\in S_k} M\left(x, n_k, \varepsilon/2^{k+1}\right)\right)$$

Let

$$D_k = \left\{y = y(x_1, ..., x_{N_k}) : (x_1, ..., x_{N_k}) \in S_k^{N_k}\right\}, \tag{7}$$

and

$$t_k = a_{N_k} + n_k = N_k n_k + N_{k-1} \times \max_{x\in S_k} M\left(x, n_k, \varepsilon/2^{k+1}\right). \tag{8}$$

The sequence $\{\ell_k\}$ is recursively defined as $\ell_1 = n_1$, and

$$\ell_{k+1} = \ell_k + \max_{x\in S_k} M\left(x, \ell_k, \varepsilon/2^{k+1}\right) + t_{k+1}.$$

Finally is introduced the sequence $\{L_k\}$ by $L_1 = D_1$, if $x \in L_k$, $y \in D_{k+1}$ then, by the non-uniform specification property, there is a $\overline{z} = \overline{z}(x,y)$ such that

$$d_{\ell_k}(x, \overline{z}) < \varepsilon/2^{k+2}, \tag{9}$$

and

$$d_{t_{k+1}}\left(f^{\ell_k+\max_{x\in L_k} M\left(x,\ell_k,\varepsilon/2^{k+1}\right)}\left(\overline{z}\right),y\right). \tag{10}$$

Thus $L_{k+1}=\{\overline{z}\left(x,y\right):x\in L_k,\ y\in D_{k+1}\}$ .if $x\in L_k,\ y\in D_{k+1}$ and $y_1,y_2\in D_{k+1}$ then

$d_{\ell_k}\left(\overline{z}\left(x,y_1\right),\overline{z}\left(x,y_2\right)\right)<\varepsilon/2^k$ and $d_{\ell_{k+1}}\left(\overline{z}\left(x,y_1\right),\overline{z}\left(x,y_2\right)\right)\ >2\varepsilon$. So that each L_k is $\left(\ell_k,2\varepsilon\right)-$separated.

The sequence $\{N_k\}$ verifies

$$N_k\geq 2^{\sum\limits_{i=1}^{k-1}N_in_i+(N_i-1)\max_{x\in S_i} M\left(x,n_i,\varepsilon/2^{i+1}\right)+\max_{x\in L_i} M\left(x,\ell_i,\varepsilon/2^{i+3}\right)+\max_{x\in S_{k+1}} M\left(x,n_{k+1},\varepsilon/2^{k+2}\right)}$$

A fractal set $F=F\left(\{n_k\},\{N_k\},\varepsilon,\{S_k\}\right)$ is defined as

$$F=\bigcap_{k=1}^{\infty}F_k,$$

with $F_k=\bigcup\limits_{x\in L_k}B_{\ell_k}\left(x,\varepsilon/2^{k-1}\right)$.

For each $n\in\mathbf{N}$ and $x\in L_k$, let j be the unique number such that

$$\ell_k+j\left(\max_{x\in S_k}M\left(x,n_k,\varepsilon/2^{k+1}\right)+n_{k+1}\right)\leq n<\ell_k+(j+1)\left(\max_{x\in S_k}M\left(x,n_k,\varepsilon/2^{k+1}\right)+n_{k+1}\right). \tag{11}$$

It can defined a sequences of measures concentrated on F_k by

$$\mu_k=\frac{1}{A_k}\nu_k,$$

with $\nu_k=\sum\limits_{x\in L_k}\delta_x$ and $A_k=cardL_k=M_1^{N_1}...M_k^{N_k}$, where $M_k=cardS_k$. Let $\mathcal{B}=B_{n,\varepsilon/2}\left(x\right)$ such that $\mathcal{B}\cap F\neq\varnothing$, so

$$\mu_k\left(\mathcal{B}\right)\leq\frac{M_{k+1}^{N_{k+1}-j}}{M_1^{N_1}...M_k^{N_k}M_{k+1}^{N_{k+1}}}=\frac{1}{cardL_k\times M_{k+1}^{j}}.$$

Let μ be the w^*-limit of the sequence$\{\mu_k\}$, then by the distribution mass principle

$$h_{top}(F)\geq\liminf_{n\to\infty}\frac{1}{n}\left(\sum_{i=1}^{k}N_i\log M_i+j\log M_{k+1}\right). \tag{12}$$

Proposition 3.1: The fractal F is contained in the set of genereric points $G\left(\mu\right)$.

Proof: Let $\Theta=\{\varphi_1,\varphi_2,...\}$ and recall that $G\left(\mu\right)=\left\{x:\lim_{n\to\infty}\sum\limits_{i=1}^{\infty}p_i\left|S_n\left(\varphi_i\left(x\right)\right)-\int\varphi_id\mu\ \right|=0\right\}$, where $\{p_i\}$ is a sequence of numbers with $\sum_{i=1}^{\infty} p_i = 1$. Let $var\left(\varphi_i, \varepsilon\right) = \sup\left\{\left|\varphi_i\left(x\right) - \varphi_i\left(y\right)\right| : d\left(x, y\right) < \varepsilon\right\}$, if $d_n\left(x, y\right) < \varepsilon$ then

$$\left|\sum_{j=0}^{n-1} \varphi_i\left(f^j(x)\right) - \sum_{j=0}^{n-1} \varphi_i\left(f^j(y)\right)\right| \leq nvar\left(\varphi_i, \varepsilon\right).$$

Let us suppose firstly that $y \in D_k$, and estimate

$$\left|\sum_{j=0}^{t_k-1} \varphi_i\left(f^j(y)\right) - t_k \alpha_i\right|,$$

where $\alpha = \left(\alpha_i = \int \varphi_i d\mu\right)_i$. Let us consider the sets S_k in the construction of the sets D_k and L_k contained in the set $X_\Theta\left(\alpha, \delta_k, n_k\right) = \left\{x : \sum_{i=1}^{\infty} p_i \left|S_{n_k}\left(\varphi_i\left(x\right)\right) - \ \alpha_i\right| < \delta_k\right\}$, with $\delta_k \to 0$. If $y \in D_k$ then there are points $x_{\ell_j}^k \in S_k$, $j = 1, 2, ..., N_k$ such that

$$d_{n_k}\left(x_{\ell_j}^k, f^{a_m}\left(y\right)\right) < \varepsilon/2^k,$$

with

$$a_m = (m-1)\left(n_k + \max_{x \in S_k} M\left(x, n_k, \varepsilon/2^k\right)\right).$$

We have

$$\left|\sum_{j=0}^{n_k-1} \varphi_i\left(f^j(x_{\ell_j}^k)\right) - \sum_{j=0}^{n_k-1} \varphi_i\left(f^{j+a_m}(y)\right)\right| \leq n_k var\left(\varphi_i, \varepsilon/2^k\right). \tag{13}$$

Since $S_k \subset X_\Theta\left(\alpha, \delta_k, n_k\right)$ holds

$$\left|\sum_{j=0}^{n_k-1} \varphi_i\left(f^{j+a_m}(y)\right) - \alpha_i\right| \leq n_k\left(var\left(\varphi_i, \varepsilon/2^k\right) + \delta_k\right). \tag{14}$$

Set

$$[0, t_k - 1] = \bigcup_{m=0}^{N_k-1} [a_m, a_m + N_k - 1] \cup \bigcup_{m=0}^{N_k-2} \left[a_m + n_k, a_m + n_k \ \max_{x \in S_k} M\left(x, n_k, \varepsilon/2^k\right) - 1\right].$$

Thus

$$\left|\sum_{j=0}^{n_k-1} \varphi_i\left(f^{j+a_m+n_k}(y) - n_k \alpha_i\right| \leq \max_{x \in S_k} M\left(x, n_k, \varepsilon/2^k\right)\left[\left|\alpha_i\right| + \left\|\varphi_i\right\|_0\right] \tag{15}$$

So that, by eq. (15)

$$\left|\sum_{j=0}^{t_k-1} \varphi_i\left(f^{j+a_m}(y)\right) - t_k \alpha_i\right| \leq \tag{16}$$

$$N_k n_k\left(var\left(\varphi_i, \varepsilon/2^k\right) + \delta_k\right) + 2\left(N_k - 1\right) \max_{x \in S_k} M\left(x, n_k, \varepsilon/2^k\right) \left\|\varphi_i\right\|_0 \tag{17}$$

.

The next step in to get an estimation on L_k. Let

$$R_{k,i} = \max_{z \in L_k} \left\{ \left| \sum_{j=0}^{\ell_k - 1} \varphi_i \left(f^j(z) - \ell_k \alpha_i \right| \right\},$$

so is valid $R_{1,i} \leq \ell_1 \delta_1$, for any i. Let $x \in L_k$, $y \in D_{k+1}$, $z \in L_{k+1}$, by the construction the sets

$$d_{\ell_k} (x, z) < \varepsilon / 2^{k+1},$$

and

$$d_{t_{k+1}} \left(f^{\ell_k + \max_{x \in L_k} M\left(x, \ell_k, \varepsilon/2^{k+1}\right)} (z), y \right) < \varepsilon / 2^{k+1}.$$

Thus

$$\left| \sum_{j=0}^{\ell_{k+1}-1} \varphi_i \left(f^j(z) - \ell_{k+1} \alpha_i \right| \leq \left| \sum_{j=0}^{\ell_k - 1} \varphi_i \left(f^j(z) - \sum_{j=0}^{\ell_k - 1} \varphi_i \left(f^j(x) \right| + \right.\right.$$

$$\left| \sum_{j=0}^{\ell_k - 1} \varphi_i \left(f^j(x) - \ell_k \alpha_i \right| + \left| \sum_{j=\ell_k}^{\ell_k + \max_{x \in L_k} M\left(x, \ell_k, \varepsilon/2^{k+1}\right) - 1} \varphi_i \left(f^j(z) - \max_{x \in L_{k+1}} M \left(x, \ell_k, \varepsilon/2 \right.\right.\right.$$

$$\left| \sum_{j=0}^{\ell_{k+1}-1} \varphi_i \left(f^{j + \max_{x \in L_{k+1}} M\left(x, \ell_k, \varepsilon/2^{k+2}\right) + \ell_k} (z) \right) - \sum_{j=0}^{\ell_{k+1}-1} \varphi_i \left(f^j(y) \right| + \right.$$

$$\left| \sum_{j=0}^{\ell_{k+1}-1} \varphi_i \left(f^j(y) - t_{k+1} \alpha_i \right| \leq \ell_k var \left(\varphi_i, \varepsilon/2^{k+1} \right) + R_{k,i} + 2 \max_{x \in L_{k+1}} M \left(x, \ell_k, \varepsilon/2^{k+2} \right) \|\varphi_i\|_0 + t_{k+1} var \left(\varphi_i, \varepsilon/2^{k \right.\right.$$

$$N_{k+1} n_{k+1} \left(var \left(\varphi_i, \varepsilon/2^{k+1} \right) + \delta_{k+1} \right) + 2 \left(N_{k+1} - 1 \right) \max_{x \in L_{k+1}} M \left(x, \ell_k, \varepsilon/2^{k+2} \right) \|\varphi_i\|_0 .$$

$$R_{k,i} \leq 2 \sum_{j=1}^{k} \ell_j \left(var \left(\varphi_i, \varepsilon/2^j \right) + \delta_j + \frac{N_j \max_{x \in L_j} M \left(x, \ell_k, \varepsilon/2^j \right)}{\ell_j} \|\varphi_i\|_0 . \right).$$

Since X is compact $\lim\limits_{\varepsilon \to 0} var \left(\varphi_i, \varepsilon \right) = 0$, also $\delta_k \to 0$. We may choose the sequence$\{n_k\}$with $n_k \to \infty$ such that $n_k \geq 2^{\max_{x \in L_k} M(x, \ell_k, \varepsilon/2k)}$ so we can express $R_{k,i}$ bounded as

$$R_{k,i} \leq \sum_{j=1}^{k} \ell_j T_j,$$

where $T_k \to 0$ as $k \to \infty$. So that

$$\frac{R_{k,i}}{\ell_k} \leq T_k + \frac{1}{k}\sum_{j=1}^{k} \ell_j T_j$$

and $\sum_{i=1}^{\infty} p_i R_{k,i} \leq \sum_{j=1}^{k} \ell_j T_j$. Thus, for k enough large and since $\ell_k \geq 2^{\ell_k - 1}$ we have

$$\sum_{i=1}^{\infty} p_i \; \frac{R_{k,i}}{\ell_k} \leq T_k + \frac{1}{k}\sum_{j=1}^{k} T_j. \tag{18}$$

Therefore $\sum_{i=1}^{\infty} p_i \; \frac{R_{k,i}}{\ell_k} \to 0$ as $k \to \infty$.

Finally is done the estimation on F. Let $x \in F$, $n \in \mathbf{N}$, $n > \ell_1$, there is an unique number k such that $\ell_k < n < \ell_{k+1}$. Besides there exist a number m such that $n \geq \ell_k + j\left(\max_{x \in S_k} M\left(x, n_k, \varepsilon/2^{k+1}\right) + n_{k+1}\right)$.

If $x \in F$ then there exists a point $z \in L_{k+1}$ such that

$$d_{\ell_{k+1}}\left(x, z\right) < \varepsilon/2^k.$$

If $z \in L_{k+1}$ then there exist $x \in L_k$, $y \in D_{k+1}$ such that

$$d_{\ell_k}\left(x, z\right) < \varepsilon/2^{k+1}.$$

For each $z \in L_{k+1}$ there are points $\overline{x} \in L_k$, $y \in D_{k+1}$

$$d_{\ell_k}\left(\overline{x}, z\right) < \varepsilon/2^{k+1}$$

and

$$d_{t_{k+1}}\left(f^{\ell_k + \max_{x \in L_k} M\left(x, \ell_k, \varepsilon/2^{k+1}\right)}\left(z\right), y\right) < \varepsilon/2^{k+1}.$$

Thus we have $d_{\ell_k}\left(\overline{x}, x\right) < \varepsilon/2^{k-1}$ and $d_{t_{k+1}}\left(f^{\ell_k + \max_{x \in L_k} M\left(x, \ell_k, \varepsilon/2^{k+1}\right)}\left(x\right), y\right) < \varepsilon/2^{k-1}$. If $m > 0$ there are points $x_{\ell_1}^{k+1}, ..., x_{\ell_m}^{k+1} \in S_{k+1}$, such that

$$d_{n_{k+1}}\left(x_{\ell_m}^{k+1}, f^{a_m}\left(y\right)\right) < \varepsilon/2^{k+1}$$

with

$$a_m = (m-1)\left(n_{k+1} + \max_{x \in S_{k+1}} M\left(x, n_{k+1}, \varepsilon/2^{k+1}\right)\right).$$

So that

$$d_{t_{nk+1}}\left(f^{\ell_k + \max_{x \in S_{k+1}} M\left(x, n_{k+1}, \varepsilon/2^{k+1} + a_m\right)}\left(z\right),\; x_{\ell_m}^{k+1}\right) < \varepsilon/2^{k-2}. \tag{19}$$

Let us consider the interval $[0, n-1]$ partitioned as

$$[0,n-1]=[0,\ell_{k-1}]\cup\bigcup_{i=1}^{m}\left[\ell_k+(i-1)\left(n_{k+1}+\max_{x\in S_k}M\left(x,n_k,\varepsilon/2^{k+1}\right)\right),\ell_k+i\left(n_{k+1}+\max_{x\in S_k}M\left(x,n_k,\varepsilon/2^{k}\right.\right.\right.$$
$$\left[\ell_k+m\left(n_{k+1}+\max_{x\in S_k}M\left(x,n_k,\varepsilon/2^{k+1}\right)\right),n-1\right].$$

Thus

$$\left|\sum_{j=0}^{\ell_k-1}\varphi_i\left(f^j(x)-\ell_k\alpha_i\right|\le\left|\sum_{j=0}^{\ell_k-1}\varphi_i\left(f^j(x)-\sum_{j=0}^{\ell_k-1}\varphi_i\left(f^j(\overline{x})\right|+\left|\sum_{j=0}^{\ell_k-1}\varphi_i\left(f^j(x)-\ -\ell_k\alpha_i\right|\le\right.\right.\right.$$
$$\ell_k var\left(\varphi_i,\varepsilon/2^{k+1}\right)+R_{k,i}.$$

In each interval of the form $\left[r_i,r_i+(i-1)\left(\max_{x\in S_{k+1}}M\left(x,n_{k+1},\varepsilon/2^{k+1}\right)+n_{k+1}\right)\right]$, using eq. (20) and the fact that $S_{k+1}\subset X_\Theta\left(\alpha,\delta_{k+1},n_{k+1}\right)$ it can be done the estimation

$$\left|\sum_{j=r_i}^{r_i+\max_{x\in S_{k+1}}M\left(x,n_{k+1},\varepsilon/2^{k+1}\right)+n_{k+1}-1}\varphi_i\left(f^j(x)-\left(\max_{x\in S_k}M\left(x,n_k,\varepsilon/2^{k+1}\right)+n_{k+1}\right)\alpha_i\right|\le. \quad (20)\right.$$

$$2\max_{x\in S_k}M\left(x,n_k,\varepsilon/2^{k+1}\right)\|\varphi_i\|_0+n_{k+1}var\left(\varphi_i,\varepsilon/2^{k-2}\right).$$

On the intervals $\left[\ell_k+m\left(n_{k+1}+\max_{x\in S_k}M\left(x,n_k,\varepsilon/2^{k+1}\right)\right),n-1\right]$ we have

$$\left|\sum_{s=\ell_k+m\left(n_{k+1}+\max_{x\in S_k}M\left(x,n_k,\varepsilon/2^{k+1}\right)\right)}^{n-1}\varphi_i\left(f^s(x)-\left(n-\ell_k-m\left(n_{k+1}+\max_{x\in S_{k+1}}M\left(x,n_{k+1},\varepsilon/2^{k+1}\right)\right)\right)\alpha_i\right|\le\right.$$

$$2\left(n-\ell_k-m\left(n_{k+1}+\max_{x\in S_{k+1}}M\left(x,n_{k+1},\varepsilon/2^{k+1}\right)\right)\right)\|\varphi_i\|_0\le2\left(n_{k+1}+\max_{x\in S_k}M\left(x,n_{k+1},\varepsilon/2^{k+1}\right)\right)\|\varphi_i\|_0.$$

Finally

$$\left|\sum_{j=0}^{n-1}\varphi_i\left(f^j(x)-n\alpha_i\right|\le R_{k,i}+\left(\ell_k+mn_{k+1}\right)var\left(\varphi_i,\varepsilon/2^{k-2}\right)+\right.$$

$$2\left(n_{k+1}+(m+1)\max_{x\in S_k} M\left(x,n_k,\varepsilon/2^{k+1}\right)\right)\|\varphi_i\|_0+mn_{k+1}\delta_{k+1}.$$

Recall that $n\geq \ell_k+j\left(\max_{x\in S_k} M\left(x,n_k,\varepsilon/2^{k+1}\right)+n_{k+1}\right)$ and $\ell_k>N_k$, so

$$\sum_{i=1}^{\infty} p_i\left|S_n\left(\varphi_i\left(x\right)\right)-\alpha_i\right|\leq\sum_{i=1}^{\infty} p_i\ \frac{R_{k,i}}{\ell_k} var\left(\varphi_i,\varepsilon/2^{k-2}\right)+$$

$$2\left(\frac{n_{k+1}+\max\limits_{x\in S_{k+1}} M\left(x,n_{k+1},\varepsilon/2^{k+1}\right)}{N_k}+\frac{\max\limits_{x\in S_{k+1}} M\left(x,n_{k+1},\varepsilon/2^{k+1}\right)}{n_{k+1}}\right)\|\varphi_i\|_0+\delta_{k+1}.$$

The right hand tends to 0 as $n\to\infty$ and $k\to\infty$, so that

$$\lim_{n\to\infty}\sum_{i=1}^{\infty} p_i\left|S_n\left(\varphi_i\left(x\right)\right)-\alpha_i\ \right|=0,$$

therefore $x\in G\left(\mu\right)$.

■

Proposition 3.2: For systems with nonuniform specification and for a fixed $\mu\in\mathcal{M}_{inv}(X,f)$, holds that

$$\Lambda_\Theta\left(\alpha\right)\geq h_\mu\left(f\right).$$

Proof: Let

$$var\left(\varphi_i,\varepsilon\right)=\sup\left\{\left|\varphi_i\left(x\right)-\varphi_i\left(y\right)\right|:d\left(x,y\right)<\varepsilon\right\},$$

since X is compact and $\varphi_i\in C(X)$ we have $\lim\limits_{\varepsilon\to 0} var\left(\varphi_i,\varepsilon\right)=0$. It can be chosen $\varepsilon>0$, $\delta>0$ such that

$$\sum_{i=1}^{\infty} p_i var\left(\varphi_i,\varepsilon\right)<\varepsilon<\gamma.$$

with $0<\gamma<1$ given by lemma 2.1, and

$$\lim_{n\to\infty}\sup\frac{1}{n}\log N\left(\alpha,\delta,\epsilon,n\right)<\Lambda_\Theta\left(\alpha\right)+\varepsilon.$$

Let $R_n\left(\varepsilon,\delta,\mu\right)$ be the minimal number of balls $B_{n,\varepsilon}\left(x\right)$ whose union has $\mu-$measure $\geq 1-\delta$. By a theorem of Katok[**?**] if μ is ergodic then holds

$$h_\mu(f) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \sup R_n(\varepsilon,\delta,\mu) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \inf R_n(\varepsilon,\delta,\mu).$$

Let us consider the measure ν and its convex ergodic decomposition given by the lemma 2.1 $\nu = \sum_{j=1}^{k} \lambda_j \nu_j$, thus for any $\varepsilon > 0$ there is a $\ell_j = \ell_j(\nu_i, 4\varepsilon, \gamma) \geq 1$, such that

$$R_n(4\varepsilon, \gamma,\ \nu_i) \geq \exp\left[n\left(h_{\nu_j}(f) - \gamma\right)\right], \quad \text{for } n \geq \ell_j, \quad j = 1, ..., k.$$

Let N_0 enough large such that $n_j := [\lambda_j n] \geq \max\{\ell_1, ..., \ell_k, N\}$, for $n \geq \ell_j$ and where N is sush that $\nu_j(Y_j(N)) > 1 - \gamma$. Let E_j, $j = 1, ..., r$, be a finite $(n_j, 4\varepsilon)-$separated set in $Y_j(N)$, by the non-uniform specification property given points $x_1, x_2, ..., x_r$, $x_j \in E_j$, there is a $y = y(\ x_1, x_2, ..., x_r)$ such that

$$d_{n_j}(f^{a_j}(y), x_j) < \varepsilon/2^k,$$

with $a_j = \sum_{i=1}^{j-1}\left(n_i + \max_{y\in E_i} M\left(y, n_i, \varepsilon/2^{k+1}\right)\right)$. By [7] hold these two facts

i) $y = y(\ x_1, x_2, ..., x_r) \in X_\Theta(\alpha, 5\delta, \widehat{n})$, with $\widehat{n} = a_k + n_k$.

ii) If $y = y(\ x_1, x_2, ..., x_k)$, $\ddot{y} = \ddot{y}\left(\ \acute{x}_1, \acute{x}_2, ..., \acute{x}_k\right)$ correspond to different $r-$tuples $(\ x_1, x_2, ..., x_r)$, $\left(\ \acute{x}_1, \acute{x}_2, ..., \acute{x}_r\right)$ then $d\left(y, \ddot{y}\right) > 5\varepsilon$.

Thus is valid

$$N(\alpha, 5\delta, \epsilon, \widehat{n}) \geq M_1 ... M_k,$$

with $M_j = card E_j$.Since each E_j is $(n_j, 4\varepsilon)-$separated set in $Y_j(N)$, then

$$M_j \geq \exp\left[n_j\left(h_{\nu_j}(f) - \gamma\right)\right].$$

and

$$N(\alpha, 5\delta, \epsilon, \widehat{n}) \geq \exp\left[\sum_{j=1}^{k} [\lambda_j n]\left(h_{\nu_j}(f) - \gamma\right)\right].$$

Since $\sum_{j=1}^{k} \lambda_j = 1$ and $\frac{[\lambda_j n]}{n} \to \lambda_j$, as $n \to \infty$, we have

$$\liminf_{\widehat{n}\to\infty} \frac{1}{\widehat{n}} N(\alpha, 5\delta, \epsilon, \widehat{n}) \geq h_\mu(f) - 3\gamma$$

and by (6)

$$\Lambda_\Theta(\alpha) \geq h_\mu(f) - 4\gamma.$$

■

Proposition 3.3: For dynamical systems with the non-uniform specification property holds $h_{top}(G(\mu)) = h_\mu(f)$, for any $f-$invariant measure μ.

Proof: Let $\{S_k\}$ be the sequence of finite sets and $\{n_k\}$, $\{N_k\}$be the sequence of positive integers as in the earlier constructions. Recall (c.f. eq 14) that

$$h_{top}(F) \geq \liminf_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{k} N_i \log M_i + m \log M_{k+1}\right)$$

with $M_j = cardS_j$ and m the unique number such that

$$\ell_k + m\left(\max_{x\in S_k} M\left(x, n_k, \varepsilon/2^{k+1}\right) + n_{k+1}\right) \leq n < \ell_k + (m+1)\left(\max_{x\in S_k} M\left(x, n_k, \varepsilon/2^{k+1}\right) + n_{k+1}\right).$$

The sets S_k are$(n_k, 5\varepsilon)-$separated sets, for a fixed $\varepsilon > 0$, in $X_\Theta(\alpha, \delta_k, n_k)$, we can consider $M_k = cardS_k \geq \exp\left[n_k\left(\Lambda_\Theta(\alpha) - \gamma\right)\right]$, like in proposition 3.1. Thus, since $F \subset G(\mu)$ and by proposition 3.1

$$h_{top}(G(\mu)) \geq h_{top}(F) \geq \Lambda_\Theta(\alpha) - \gamma \geq h_\mu(f) - 5\gamma,$$

for arbitrary small γ. The inequality $h_{top}(G(\mu)) \leq h_\mu(f)$ was proved by Bowen[3].

■

V. PROOF OF THE THEOREMS 1.1 AND 1.2

The following result, appeared in [5], is very useful for the proof of the variational principle (theorem 1.2) as well as for the study of the irregular part of the spectrum (theorem 1.1)

Lemma 4.1: For any $\Phi \in C(X^r)$ and for any $\varepsilon > 0$ there is a map $\widetilde{\Phi} : X^r \to \mathbf{R}$ of the form

$$\widetilde{\Phi} = \sum_{j=1}^{n} \varphi_j^{(1)} \otimes ... \otimes \varphi_j^{(r)},$$

with $\varphi_j^{(i)} \in C(X)$ and such that $\left\|\Phi - \widetilde{\Phi}\right\|_\infty < \varepsilon$.

Also is needed

Lemma 4.2(Bowen[3]): For any $t \geq 0$ holds

$$h_{top}(\{x : \exists\mu \in V(x) \text{ with } h_\mu(f) \leq t\}) \leq t.$$

In fact the proof of the theorem 1.2 is totally similar of that of the theorem 1.1 in [5].

Let us denote

$$\mathcal{M}_{\Phi}(\alpha) = \left\{ \mu \in \mathcal{M}_{inv}(X) : \int \Phi d\mu^{\otimes r} = \alpha \right\},$$

Proof of the theorem 1.2: Let $\varepsilon > 0$ and $\widetilde{\Phi}$ be the map of the lemma 4.1, so that

$$V_{\widetilde{\Phi}}(n,x) = \sum_{j=1}^{n} \prod_{i=1}^{r} \frac{1}{n} S_n \left(\varphi_j^{(i)}(x) \right),$$

where $S_n \left(\varphi_j^{(i)}(x) \right) = \sum_{k=0}^{n-1} \varphi_j^{(i)} \left(f^k(x) \right.$. Let $x \in E_{\Phi}(\alpha)$, since X is compact there is a $\mu \in V(x)$ and a sequence $\{n_k\}$ such that $w^* - \lim_{k\to\infty} \mathcal{E}_{n_k}(x) = \mu$, .where w^*- means weak convergence. Therefore

$$\lim_{n\to\infty} V_{\widetilde{\Phi}}(n_k, x) = \int \widetilde{\Phi} d\mu^{\otimes r}.$$

We have

$$\left| \int \Phi d\mu^{\otimes r} - \alpha \right| \leq \left| \int \Phi d\mu^{\otimes r} - \int \widetilde{\Phi} d\mu^{\otimes r} \right| + \left| \int \widetilde{\Phi} d\mu^{\otimes r} - V_{\widetilde{\Phi}}(n_k, x) \right| +$$
$$\text{and } \left| V_{\widetilde{\Phi}}(n_k, x) - V_{\Phi}(n_k, x) \right| + \left| V_{\Phi}(n_k, x) - \alpha \right|,$$

where $\lim_{k\to\infty} \left(V_{\widetilde{\Phi}}(n_k, x) - V_{\Phi}(n_k, x) \right) = 0$ and $\lim_{k\to\infty} \left(V_{\Phi}(n_k, x) - \alpha \right) = 0$. Thus $\left| \int \Phi d\mu^{\otimes r} - \alpha \right| < 2\varepsilon$, and, since ε is arbitrary, $\mu \in \mathcal{M}_{\Phi}(\alpha)$. Then we have that

$E_{\Phi}(\alpha) \subset \{x : \exists \mu \in V(x) \text{ with } h_{\mu}(f) \leq \sup \{h_{\mu}(f) : \mu \in \mathcal{M}_{\Phi}(\alpha)\}\}$. Hence by the Bowen lemma

$$h_{top}(E_{\Phi}(\alpha)) \leq \sup \left\{ h_{\mu}(f) : \int \Phi d\mu^{\otimes r} = \alpha \right\}.$$

To prove the opposite inequality, let $x \in G(\mu)$, with $\mu \in \mathcal{M}_{\Phi}(\alpha)$, so that

$$\lim_{n\to\infty} V_{\widetilde{\Phi}}(n, x) = \int \widetilde{\Phi} d\mu^{\otimes r}.$$

we have

$$\left| \lim_{n\to\infty} V_{\Phi}(n, x) - \int \Phi d\mu^{\otimes r} \right| \leq \left| \lim_{n\to\infty} V_{\Phi}(n, x) - \lim_{n\to\infty} V_{\widetilde{\Phi}}(n, x) \right| +$$

$$\left|\lim_{n\to\infty} V_{\widetilde{\Phi}}(n,x) - \int \widetilde{\Phi} d\mu^{\otimes r}\right| + \left|\int \widetilde{\Phi} d\mu^{\otimes r} - \int \Phi d\mu^{\otimes r}\right| < 2\varepsilon.$$

Thus $\lim_{n\to\infty} V_{\Phi}(n,x) = \int \Phi d\mu^{\otimes r} = \alpha$, since $\mu \in \mathcal{M}_{\Phi}(\alpha)$. In this way is proved that $G(\mu) \subset E_{\Phi}(\alpha)$, from this and proposition 3.3 is obtained

$$h_{top}(E_{\Phi}(\alpha)) \geq h_{top}(G(\mu)\ ) \geq h_{\mu}(f),$$

then taken sup over the measures $\mu \in \mathcal{M}_{\Phi}(\alpha)$ results

$$h_{top}(E_{\Phi}(\alpha)) \geq \sup\left\{h_{\mu}(f) : \int \Phi d\mu^{\otimes r} = \alpha\right\}.$$

■

Let

$$G_{\Phi}(\alpha) = \left\{x : \text{ there is } \{n_k\} \text{ such that } w^* - \lim_{k\to\infty} \mathcal{E}_{n_k}(x) = \mu \in \mathcal{M}_{\Phi}\ (\alpha)\right\},$$

For $\alpha_1 \neq \alpha_2 \in \mathbf{R}$, we shall find a set $G \subset G_{\Phi}(\alpha_1) \cap G_{\Phi}(\alpha_2)$.

Before proving the theorem 1.1 we give some lemmas.

Lemma 4.3: If $\alpha_1 \neq \alpha_2$ then $G_{\Phi}(\alpha_1) \cap G_{\Phi}(\alpha_2) \subset E_{\Phi}^{\infty}$.

Proof: Let $\varepsilon > 0$ and $\widetilde{\Phi}$ be the map of the lemma 4.1. Let $x \in G_{\Phi}(\alpha_1) \cap G_{\Phi}(\alpha_2)$, so there are sequences $\{n_k\}, \{m_k\}$ such that

$$\mu = w^* - \lim_{k\to\infty} \mathcal{E}_{n_k}(x)\,; \mu \in \mathcal{M}_{\Phi}(\ \alpha_1)$$
$$\nu = w^* - \lim_{k\to\infty} \mathcal{E}_{m_k}(x)\,; \nu \in \mathcal{M}_{\Phi}(\ \alpha_2),$$

We have

$$V_{\widetilde{\Phi}}(n,x) = \sum_j \prod_{i=1}^{r} \frac{1}{n} S_n\left(\varphi_j^{(i)}(x)\right),$$

where $S_n\left(\varphi_j^{(i)}(x)\right) = \sum_{k=0}^{n-1} \varphi_j^{(i)}\left(f^k(x)\right)$. Therefore

$$\lim_{k\to\infty} V_{\widetilde{\Phi}}(n_k, x) = \int \widetilde{\Phi} d\mu^{\otimes r}$$
$$\lim_{k\to\infty} V_{\widetilde{\Phi}}(m_k, x) = \int \widetilde{\Phi} d\nu^{\otimes r}.$$

By the argument of approximation of lemma 4.1 we get in the same way of [5] that

$$\lim_{k\to\infty} V_{\Phi}(n_k, x) = \int \Phi d\mu^{\otimes r} = \alpha_1 \text{ and } \lim_{k\to\infty} V_{\Phi}(m_k, x) = \int \Phi d\nu^{\otimes r} = \alpha_2, \text{ with } \alpha_1 \neq \alpha_2.$$

Then $x \in E_{\Phi}^{\infty}$.

■

For $\rho_1, \rho_2, ..., \rho_k \in \mathcal{M}(X)$ and positive numbers $R_1, R_2, ..., R_k$, let $x_1, x_2, ..., x_k \in X$, $n_1, n_2, ..., n_k \in \mathbf{N}$ such that $\mathcal{E}_{n_j}(x_j) \in B_{R_j}(\rho_j)$, $j = 1, 2, ..., k.$, for a given $\rho_1, \rho_2, ..., \rho_k \in \mathcal{M}(X)$ and $R_1, R_2, ..., R_k$. Let $\varepsilon_1 > 0$, $\varepsilon_2 > 0, ..., \varepsilon_k > 0$, by the non-uniform specification property if $M_j \geq M(x_j, n_j, \varepsilon_j)$, $j = 1, 2, ..., k$, then there is a point $z \in X$ such that

uch that

$$d_{n_1}(x_1, z) < \varepsilon$$

and

$$d_{n_i}\left(f^{\sum_{j=1}^{i-1}(n_j+M_j)}(z), x_i\right) < \varepsilon.$$

Let $s_j = n_j + M_j$, $j = 1, 2, ..., k$, and $S_j = s_1 + s_2 + ... + s_j$.

Lemma 4.4: Let z such that $d_{n_i}\left(f^{\sum_{j=1}^{i-1}(n_j+M_j)}(z), x_i\right) < \varepsilon$, then for any $\rho \in \mathcal{M}(X)$ holds

$$D\left(\mathcal{E}_{S_k}(z), \rho\right) \leq \frac{1}{S_k} \sum_{j=1}^{k} s_j \left(\overline{R_j} + D(\rho_j, \rho)\right),$$

where $\overline{R_j} = R_j + \varepsilon_j$, $j = 1, 2, ..., k$.

Proof: We have

$$\mathcal{E}_{S_k}(z) = \frac{1}{M_k} \sum_{j=1}^{k} s_j \mathcal{E}_{s_j}\left(f^{S_{j-1}}(z)\right),$$

and

$$D(\mathcal{E}_{s_j}(x_j), \mathcal{E}_{s_j}\left(f^{S_{j-1}}(z)\right)) \le \frac{1}{s_j} \sum_{l=0}^{n_j-1} d\left(f^l(x_j), f^{-S_{j-1}-l}(z)\right).$$

Therefore

$$D\left(\mathcal{E}_{S_k}(z), \rho\right) \le \frac{1}{S_k} \sum_{j=1}^{k} \left[D(\mathcal{E}_{s_j}(x_j),_j, \mathcal{E}_{n_{sj}}\left(f^{M_{j-1}}(z)\right)) + D(\mathcal{E}_{s_j}(x_j), \rho_j) + D(\rho_j, \rho)\right] \le$$
$$\frac{1}{M_k} \sum_{j=1}^{k} \left[R_j + \varepsilon_j + D(\rho_j, \rho)\right].$$

■

.

Lemma 4.5: Let $\alpha_1 \ne \alpha_2$ with $\mathcal{M}_\Phi(\alpha_1) \ne \varnothing$, $\mathcal{M}_\Phi(\alpha_2) \ne \varnothing$ then

$$h_{top}(G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)) = \min\left\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\right\}.$$

Proof: Since $G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2) \subset G_\Phi(\alpha_1)$ and $G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2) \subset G_\Phi(\alpha_2)$, by the monotonicity of the entropy we have

$$h_{top}(G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)) \le \min\left\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\right\}$$

. To prove the other inequality we shall find a set $G \subset G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)$ with $h_{top}(G) \ge \min\left\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\right\}$.

To construct G, let us choose sequences $\{n_k\}$, $\{R_k\}$, $\{\varepsilon_k\}$ with $R_k \searrow 0$ and $\varepsilon_k \searrow 0$ and, for a given sequence $\{\rho_1, \rho_2, ..., \rho_k\} \subset \mathcal{M}(X)$, for ,$\overline{\varepsilon} > \varepsilon_1$.let us consider $(n_k, \overline{\varepsilon})$ −sets $\Gamma_k \subset \{x : \mathcal{E}_{n_k}(x) \in B_{R_k}(\rho_k)\}$, so that (by the Lemma 4.4)

$$x \in \Gamma_k, z \in B_{s_k,\varepsilon_k}(x) \Longrightarrow \mathcal{E}_{s_k}(z) \in B_{R_k+\varepsilon_k}(\rho_k).$$

Let us choose now a strictly increasing sequence $\{N_k\}$ such that

$$s_{k+1} \leq R_k \sum_{j=1}^{k} s_j N_j$$

and

$$\sum_{j=1}^{k-1} s_j N_j \leq R_k \sum_{j=1}^{k} s_j N_j.$$

We consider stretched sequences $\left\{ s_j' \right\}, \left\{ \varepsilon_j' \right\}, \left\{ \Gamma_j' \right\}$ such that if $j = N_1 + ... + N_{k-1} + q$ with $1 \leq q \leq N_k$ then $n_j' = n_k, \quad \varepsilon_j' = \varepsilon_k$ and $\Gamma_j' = \Gamma_k$.

Finally, we can define

$$G_k := \bigcap_{j=1}^{k} \left(\bigcup_{x_j \in \Gamma_j'} f^{-S_{j-1}} \left(B_{s_j', \varepsilon_j'} \left(x_j \right) \right) \right),$$

with $S_j = s_1' + s_2' + ... + s_j'$ and

$$G := \bigcap_{k \geq 1} G_k.$$

Any element of G can be labelled by a sequence $x_1 \ x_2 ...,$ with $x_j \in \Gamma_j'$. According to Pfister and Sullivan [12] the following holds: Let $x_j, y_j \in \Gamma_j', x_j \neq y_j$, if $x \in B_{s_j, \epsilon_j} \left(x_j \right)$, $y \in B_{s_j, \epsilon_j} \left(y_j \right)$ then $\max \left\{ d \left(f^k(x), f^k(y) \right) : k = 0, ..., n_j - 1 \right\} > 2\varepsilon$, with $\varepsilon > \varepsilon_1 / 4$.

We see that $G \subset G_\Phi \left(\alpha_1 \right) \cap G_\Phi \left(\alpha_2 \right)$. Let $z \in G$, and let $\mu_0 \in \mathcal{M}_\Phi \left(\alpha_1 \right), \nu_0 \in \mathcal{M}_\Phi \left(\alpha_2 \right)$, it can be considered sequences[**?**] $\left\{ \mu_k \right\}, \left\{ \nu_k \right\}$ such that

$D \left(\mu_0, \mu_k \right) < R_k$ and $D \left(\nu_0, \nu_k \right) < R_k$, then form the sequence

$$\left\{ \rho_k \right\} = \left\{ \mu_1, \mu_1, \nu_1, \nu_1, \mu_2, \mu_2, \nu_2, \nu_2, ... \right\}.$$

Let $\rho \in \left\{ \mu_0, \nu_0 \right\}$, and $\sum_{l=1}^{j} s_l N_l \leq S_k \leq \sum_{l=1}^{j+1} s_l N_l$, thus

$$D \left(\mathcal{E}_{S_k} \left(z \right), \rho \right) \leq \frac{1}{S_k} \sum_{l=1}^{j-1} s_l N_l D \left(\mathcal{E}_{\sum\limits_{l=1}^{j-1} s_l N_l} \left(z \right), \rho \right) + \frac{s_j N_j}{S_k} D \left(\mathcal{E}_{s_j N_j}(z), \rho \right) +$$

$$\frac{S_k - \sum_{l=1}^{j} s_l N_{lj}}{S_k} D \left(\mathcal{E}_{s_{j+1} N_{j+1}}(z), \rho \right).$$ Therefore

$$D \left(\mathcal{E}_{S_k} \left(z \right), \rho \right) \leq R_j + D \left(\mathcal{E}_{s_j N_j}(z), \rho_j \right) + D \left(\rho_j, \rho \right) + D \left(\mathcal{E}_{s_{j+1} N_{j+1}}(z), \rho \right) + D \left(\rho_{j+1}, \rho \right) \leq$$

$2R_j+\varepsilon_j+D(\rho_j,\rho)+D(\rho_{j+1},\rho)$. Thus, choosing subsequences $t_k=4k+1$ and $u_k=4k+3$, we get

$$\mu_0 = w^* - \lim_{k\to\infty} \mathcal{E}_{S_{t_k}}(z)$$
$$\nu_0 = w^* - \lim_{k\to\infty} \mathcal{E}_{S_{u_k}}(z),$$

so that $z\in G_\Phi(\alpha_1)\cap G_\Phi(\alpha_2)$.

To complete the proof it must be proved that $h_{top}(G) \geq \min\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\}$, for this we follow [12]. Let $s < \overline{h} := \min\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\}$, the set G is closed, and so it is compact, let us consider a finite covering $\mathcal{U}$ by balls $B_{m,\varepsilon}(x)$ having non-empty intersection with G. Now

$$M(G,s,N,\varepsilon) = \inf_{\mathcal{U}\in\mathcal{C}(n,\varepsilon,G)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{U}} \exp(-sm).$$

For any finite covering $\mathcal{U}$ of G, we can construct a covering $\mathcal{U}_0$ in the following way: each ball $B_{m,\varepsilon}(x)$ is replaced by a ball $B_{M_{rr},\varepsilon}(x)$ with $M_r\leq m\leq M_{r+1}$. Thus

$$M(G,s,N,\varepsilon) = \inf_{\mathcal{U}\in\mathcal{C}(n,\varepsilon,G)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{U}} \exp(-sm) \geq \inf_{\mathcal{U}\in\mathcal{C}(N,\varepsilon,G)} \sum_{B_{M_r,\varepsilon}\in\mathcal{U}_0} \exp(-sM_{r+1}).$$

Now we can consider a covering $\mathcal{U}_0$ in which $m=\max\{r : \text{there is a ball } B_{M_r,\varepsilon}(x)\in\mathcal{U}_0\}$. We set

$$W_k := \prod_{i=1}^{k}\Gamma_i,\quad \overline{W_m} = \bigcup_{k=1}^{m} W_k.$$

Let $x_j,\ y_j\in\Gamma'_j,\ x_j\neq y_j$, as we pointed out earlier, if $x\in B_{N'_j,\varepsilon'_j}(x_j)$, $y\in B_{N'_j,\varepsilon'_j}(y_j)$ then $d\left(f^l(x),f^l(y)\right)>2\varepsilon$

for any $l=0,...,N_j-1$, and with $\varepsilon>\varepsilon_1/4$. Now for any $x\in B_{M_r,\varepsilon}(z)\cap G$ there is a, uniquely determined $z=z(x)\in W_r$. A word $\overline{w}\in W_j$, with $j=1,2,...,k$, is a called a prefix of a word $w\in W_k$ if the first $j-$letters of $\overline{w}$ agree with the first $j-$letters of w.The number of times that each $w\in W_k$ is a prefix of a word in W_m is

$cardW_m/cardW_k$, thus if W is a subset of $\overline{W_m}$ then

$$\sum_{k=1}^{m} \frac{card\,(W \cap W_k)}{card\,(W_k)} \geq card\,(W_m)\,.$$

If each word in W_m has a prefix contained in a $W \subset \overline{W_m}$ then

$$\sum_{k=1}^{m} \frac{card\,(W \cap W_k)}{card\,(W_k)} \geq 1,$$

and since $\mathcal{U}_0$ is a covering each point of W_m has a prefix associated to a ball in $\mathcal{U}_0$. By this and because $cardW_k \geq \exp\left(\overline{h}M_r\right)$, we obtain

$$\sum_{B_{M_r,\varepsilon} \in \mathcal{U}_0} \exp\left(-sM_r\right) \geq 1.$$

Thus if r is taken such that $k \geq r$ then $sM_{k+1} \leq \overline{h}M_k$, for $N \geq M_r$, $\mathcal{U} \in \mathcal{G}\,(N, \varepsilon, G)\,.$

Therefore

$$\sum_{B_{m,\varepsilon}(x) \in \mathcal{U}} \exp\left(-sm\right) \geq 1,$$

and so

$$M\,(G, s, N, \varepsilon) \geq 1.$$

By this $h_{top}(G) \geq \overline{h}$.

■

Proof of the theorem 1.1.: Let

$$\Psi = \Psi_{r,\Phi} : \mathcal{M}(X) \to \mathbf{R}$$
$$\Psi\,(\mu) = \int \Phi d\mu^{\otimes r}$$

and let

$h = h_{top}(X)$ be the topological entropy of the whole space X. By the classical variational principle and by the variational principle of [5]

$$h=\sup\left\{h_{\mu}\left(f\right):\mu\in\mathcal{M}_{inv}(X,f)\right\}=\sup_{\alpha\in\mathrm{Im}(\Psi)}\left\{h_{\mu}\left(f\right):\mu\in\mathcal{M}_{\Phi}\left(\alpha\right)\ \right\}=$$

$$\sup_{\alpha\in\mathrm{Im}(\Psi)}\left\{h_{top}(E_{\Phi}\left(\alpha\right))\ \right\}.$$

We must show that $h_{top}(E_{\Phi}^{\infty})\ \geq h$. For any $\gamma>0$, there is an $\alpha_1\in\mathrm{Im}\,\Psi$ such that $h_{top}(E_{\Phi}\left(\alpha_1\right))>h-\gamma$, let $\alpha_2\in\mathrm{Im}\,\Psi$ and let $\mu_1,\,\mu_2\in\mathcal{M}(X,f)$ with $\Psi\left(\mu_1\right)=\alpha_1,\,\Psi\left(\mu_2\right)=\alpha_2$. The map $\lambda\longmapsto\Psi\left(\left(1-\lambda\right)\mu_1+\lambda\mu_2\right)$ is continuous.Recall that

$$h_{top}(G_{\Phi}\left(\alpha_1\right)\cap\ G_{\Phi}\left(\left(1-\lambda\right)\alpha_1+\lambda\alpha_2\right))=\min\left\{h_{top}(G_{\Phi}\left(\alpha_1\right),h_{top}\left(\ G_{\Phi}\left(\left(1-\lambda\right)\alpha_1+\lambda\alpha_2\right)\right)\right\},$$

then, by the continuity of Ψ as a function of λ, we have

$$h_{top}(E_{\Phi}^{\infty})\ \geq\lim_{\lambda\rightarrow0}\ h_{top}(G_{\Phi}\left(\alpha_1\right)\cap G_{\Phi}\left(\left(1-\lambda\right)\alpha_1+\lambda\alpha_2\right))\geq$$

$$h_{top}(G_{\Phi}\left(\alpha_1\right)\geq h_{top}(E_{\Phi}\left(\alpha_1\right))>h-\gamma.$$

Since γ is arbitrary the result follows.

VI.

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