Entanglement of finite cyclic chains at factorizing fields

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We examine the entanglement of cyclic spin-1/2 chains with anisotropic *XYZ* Heisenberg couplings of arbitrary range at transverse factorizing magnetic fields. At these fields the system exhibits a degenerate symmetry-breaking separable ground state (GS). It is shown, however, that the side limits of the GS pairwise entanglement at these fields are actually *nonzero* in finite chains, corresponding such fields to a GS spin-parity transition. These limits exhibit universal properties like being independent of the pair separation and interaction range, and are directly related to the magnetization jump. Illustrative exact results are shown for chains with (I) full range and (II) nearest-neighbor couplings. Global entanglement properties at such points are also discussed.

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Quantum entanglement is well-recognized as a fundamental resource in quantum-information science [1]. It also provides a new perspective for the analysis of quantum many-body systems, allowing one to identify the genuine quantum correlations [2–4]. An important result for quantum spin chains with finite range interactions is that in contrast with the correlation length, the *pairwise* entanglement range does not necessarily diverge at a quantum phase transition [2]. For instance, it remains confined to just first and second neighbors in a nearest-neighbor Ising chain placed in a transverse magnetic field [2]. It can, however, diverge at a different point. Spin chains with anisotropic coupling exhibit a remarkable *factorizing field* [4–6], where the system possesses a *separable* ground state (GS) and hence entanglement vanishes in principle, although it was shown to reach *infinite range* in its vicinity [7].

Previous analyses were focused on large systems. The purpose of this work is to investigate the entanglement of *finite* chains, relevant for quantum-information processing, with anisotropic *XYZ* coupling of *arbitrary* range, exactly *at* transverse factorizing fields. Although it may seem that any type of entanglement will vanish at such points, it should be noticed that separable ground states break a fundamental symmetry of these chains (the S_z parity or global phase flip [2]) and are hence degenerate (and nonorthogonal), the factorizing field corresponding actually to a GS transition between opposite parity states. As a consequence, pairwise entanglement in finite chains will be shown to approach *distinct nonzero side limits* at the factorizing field, which do not depend on the pair separation or coupling range and whose average is directly measurable through the magnitude of the magnetization jump at the transition. Moreover, even the projector onto the GS subspace remains entangled at these fields. Although these effects become negligible in large anisotropic chains, where opposite parity ground states are nearly degenerate, they will be shown to be quite prominent in small finite chains and, moreover, to remain appreciable for increasing sizes if the *XY* anisotropy becomes sufficiently small. Present results are therefore particularly relevant for chains close to the *XXZ* limit. We derive first general exact results valid for any range, describing then illustrative exact results for chains with full range and nearest-neighbor couplings. The exact results in the last case are obtained through the Jordan-Wigner mapping and its analytic parity dependent diagonalization.

We consider a cyclic chain of n qubits or spins interacting through an *XYZ* Heisenberg coupling with arbitrary common range in a transverse magnetic field b. Denoting with $\boldsymbol{s}^i$ the spin at site i, the Hamiltonian reads

$$\begin{aligned} H &= bS_z - \sum_{i<j} r_{j-i}(v_x s_x^i s_x^j + v_y s_y^i s_y^j + v_z s_z^i s_z^j) \\ &= bS_z - \sum_{i<j} r_{j-i}\left[\frac{1}{2}(v_+ s_+^i s_-^j + v_- s_+^i s_+^j + \text{H.c.}) + v_z s_z^i s_z^j\right], \end{aligned} \tag{1}$$

where $\boldsymbol{S}=\sum_{i=1}^n \boldsymbol{s}^i$, $v_\pm=(v_x \pm v_y)/2$, and $r_l = r_{n-l}$ for $l=j-i = 1,\ldots,n-1$. Without loss of generality we can here assume $b \geq 0$ and $v_x \geq |v_y|$ (i.e., $v_\pm \geq 0$), with r_l arbitrary. We will be interested in the attractive (ferromagnetic) case $r_l \geq 0\ \forall\ l$, although the following considerations are general. Since H conserves the S_z-parity,

$$[H,P_z]=0, \quad P_z=\exp[i\pi(S_z+n/2)], \tag{2}$$

its nondegenerate eigenstates will have definite parity $P_z = \pm 1$.

Let us now examine the conditions for which a completely symmetric separable state of the form

$$|\theta\rangle = \prod_{i=1}^{n}\left(\cos\frac{1}{2}\theta|\downarrow_i\rangle + \sin\frac{1}{2}\theta|\uparrow_i\rangle\right) = \exp[i\theta S_y]|0\rangle, \tag{3}$$

where $s_z^i|\downarrow_i\rangle = -\frac{1}{2}|\downarrow_i\rangle$ and $|0\rangle = \Pi_i|\downarrow_i\rangle$, can be an *exact* eigenstate of Eq. (1). This state is fully aligned along an axis z' forming an angle θ with the z axis, such that $S_{z'}|\theta\rangle = -\frac{1}{2}n|\theta\rangle$, *breaking parity symmetry* for $\theta \in (0,\pi)$. Replacing $s_{z,x}^i = s_{z',x'}^i \cos\theta \pm s_{x',z'}^i \sin\theta$ in Eq. (1), it is easily seen that these conditions are

$$\cos\theta = \pm\sqrt{\chi}, \quad \chi \equiv \frac{v_y - v_z}{v_x - v_z}, \tag{4}$$

$$b = r(v_x - v_z)\cos\theta, \quad r \equiv \frac{1}{2}\sum_{l=1}^{n-1} r_l, \tag{5}$$

where Eq. (5) is required for $\theta \in (0,\pi)$, i.e., $\chi \in [0,1)$ [in the *XXZ* case $\chi=1$ ($v_y=v_x$) both $|0\rangle$ and $|\pi\rangle$ are trivial eigenstates of Eq. (1) for *all* fields b]. Such a parity breaking separable eigenstate is then feasible for $\chi \in [0,1)$ (i.e., $v_z \le v_y < v_x$ if $v_x > |v_y|$) and $b = \pm b_s$, with

$$b_s \equiv r(v_x - v_z)\sqrt{\chi}, \tag{6}$$

the *factorizing field*. The state (3) will depend on the anisotropy χ but not on the factors r_l, being then *independent* of the interaction range.

It is also apparent that *both* $|\theta\rangle$ and $|-\theta\rangle = P_z|\theta\rangle$ are degenerate eigenstates of H at $b=b_s$, with energy

$$\langle\theta|H|\theta\rangle = -\frac{1}{2}n\left[b\cos\theta + \frac{1}{2}r(v_x\sin^2\theta + v_z\cos^2\theta)\right] = -\tfrac{1}{4}nr(v_x+v_y-v_z). \tag{7}$$

Hence at $b=\pm b_s$ two levels of opposite parity necessarily cross, enabling the formation of these eigenstates.

Let us remark that in the attractive case $r_l \ge 0 \,\forall\, l$, the state that minimizes $\langle H\rangle$ among separable states (i.e., the *mean field* approximate GS) is precisely of the form (3) $\forall b$, with $|\theta|$ determined by Eq. (5) if $|b| < b_c = r(v_x - v_z)$ (parity-breaking solution) and $\theta=0$ otherwise. Hence in this case the factorizing field can be seen as that where the mean field GS becomes an *exact* eigenstate (i.e., the exact GS, as shown below).

The states $|\pm\theta\rangle$ will then form a basis of the corresponding eigenspace at $b=b_s$ (assumed of dimension 2), which is *nonorthogonal* for $\theta \ne \pi/2$: $\langle -\theta|\theta\rangle = \cos^n\theta$. A proper orthonormal basis conserving parity symmetry is provided by the *entangled* states

$$|\theta_\pm\rangle \equiv \frac{|\theta\rangle \pm |-\theta\rangle}{\sqrt{2(1\pm\cos^n\theta)}} \tag{8a}$$

$$= \sum_{k\ \substack{\text{even}\\ \text{odd}}} \frac{\sqrt{2}\sin^k\frac{\theta}{2}\cos^{n-k}\frac{\theta}{2}}{k!\sqrt{1\pm\cos^n\theta}} S_+^k|0\rangle, \tag{8b}$$

which satisfy $P_z|\theta_\pm\rangle = \pm|\theta_\pm\rangle$ *and are the actual eigenstates of* H in each parity subspace at $b=b_s$. These states (and not the states $|\pm\theta\rangle$) are the actual limits of the corresponding exact eigenstates $|\Psi^\pm(b)\rangle$ (which have definite parity) for $b \to b_s$.

In the *attractive* case $r_l \ge 0 \,\forall\, l$ (with $|v_y| \le v_x$), the states $|\theta_\pm\rangle$ (and hence $|\pm\theta\rangle$) are *ground states* of H at $b=b_s$: The exact GS $|\Psi_0^\mp(b)\rangle$ in each parity subspace must have expansion coefficients all of the *same* sign in the standard computational basis (i.e., that of separable states with definite values of $\{s_z^i\}$) in order to minimize the average energy, since the average of the off-diagonal *XY* term in Eq. (1) can only increase (or eventually stay constant) for different signs (as $r_{j-i} \ge 0$, $v_\pm \ge 0$) while those of the diagonal terms bS_z and $v_z s_z^i s_z^j$ are sign independent. Hence $|\Psi_0^\pm(b)\rangle$ cannot be orthogonal to $|\theta_\pm\rangle$, whose expansion coefficients in this basis are all nonzero and of the same sign [Eq. (8a) and (8b)], and must then coincide with $|\theta_\pm\rangle$ at $b=b_s$.

Thus in the attractive case $|\theta_\pm\rangle$ represent the *side limits* $\lim_{b\to b_s^\pm}|\Psi_0(b)\rangle$ of the exact GS $|\Psi_0(b)\rangle$ in the whole space at $b=b_s$, which undergoes there a $|\theta_-\rangle \to |\theta_+\rangle$ parity transition (actually the last parity transition as b increases, as will be shown in the examples).

The pairwise entanglement in the states $|\theta_\pm\rangle$ depends essentially on the *overlap* $\langle -\theta|\theta\rangle$. When orthogonal ($\theta=\pi/2$), they are generalized GHZ states [8], which, although globally entangled, exhibit no pairwise entanglement (for $n>2$). Moreover, in this case the normalized projector onto the space spanned by the states $|\theta_\pm\rangle$,

$$\rho_0 = \frac{1}{2}(|\theta_+\rangle\langle\theta_+| + |\theta_-\rangle\langle\theta_-|), \tag{9}$$

which represents in the attractive case the $T\to 0$ limit of the thermal mixed state $\rho(T) \propto \exp[-H/kT]$ at $b=b_s$, is fully separable (i.e., a convex combination of projectors onto separable states) as $\rho_0 = \frac{1}{2}(|\theta\rangle\langle\theta| + |-\theta\rangle\langle-\theta|)$ for $\theta=\pi/2$. In contrast, for $\theta \in (0,\pi/2)$ both states (8a) as well as the mixed state (9) will be shown to exhibit *a uniform nonzero entanglement between any two spins* (note that the projector onto this subspace is no longer the sum of the individual projectors $|\pm\theta\rangle\langle\pm\theta|$ when $\langle-\theta|\theta\rangle \ne 0$).

Let us first evaluate the pairwise concurrence [9] (a measure of pairwise entanglement) in the states $|\theta_\pm\rangle$. As a consequence of Eq. (2) and the cyclic nature of H, the reduced two spin density matrix ρ_{ij} in any nondegenerate eigenstate or in $\rho(T)$ will commute with the reduced parity $P_z^{ij} = e^{i\pi(s_z^i + s_z^j + 1)}$ and depend just on $l=|i-j|$. The ensuing concurrence $C_l \equiv C(\rho_{ij})$ takes the form

$$C_l = 2\,\mathrm{Max}[|\alpha_l^+| - p_l, |\alpha_l^-| - q_l, 0], \tag{10}$$

where $\alpha_l^\pm = \langle s_+^i s_\pm^j\rangle$, $p_l = \frac{1}{4} - \langle s_z^i s_z^j\rangle$, and $q_l = [(\frac{1}{2}-p_l)^2 - \langle s_z^i\rangle^2]^{1/2}$. If $|\alpha_l^+| > p_l$ ($|\alpha_l^-| > q_l$) C_l is of even (odd) parity type, i.e., parallel (antiparallel) [7], as in Bell state $\propto|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$ ($|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$). Just one of these inequalities can be satisfied in a given state.

In the states (8a) and (8b), $\alpha_l^\nu = \frac{1}{4}\sin^2\theta\gamma_\pm^\nu$, $p_l = \alpha_l^-$, $\langle s_z^i\rangle = -\frac{1}{2}\cos\theta\gamma_\pm^+$, with $\gamma_\pm^\nu = \frac{1\pm\nu\cos^{n-2}\theta}{1\pm\cos^n\theta}$ and $\nu=\pm$. We then obtain $C_l(|\theta_\pm\rangle) = C_\pm \,\forall\, l$, with (assuming $\theta \in (0,\pi/2]$)

$$C_\pm = \sin^2\theta \frac{\cos^{n-2}\theta}{1\pm\cos^n\theta} \tag{11a}$$

$$= (1-\chi)\frac{\chi^{n/2-1}}{1\pm\chi^{n/2}}. \tag{11b}$$

Thus $C_- > C_+ > 0$, with C_+ (C_-) *parallel* (*antiparallel*). Note that for $\theta \to 0$ ($\chi \to 1$),

$$C_+ \to 0,$$

$$C_- \to 2/n,$$

as in this limit $|\theta_+\rangle \to |0\rangle$ but $|\theta_-\rangle \to |1\rangle \equiv \frac{1}{\sqrt{n}}S_+|0\rangle$, which is a *W state* [8] ($2/n$ is in fact the *maximum* value that can be

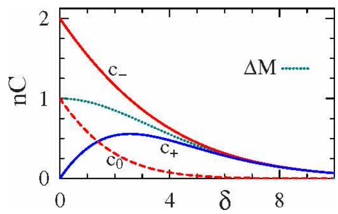

FIG. 1. (Color online) Ground state rescaled concurrences at the factorizing field vs scaled anisotropy parameter δ. $c_{\pm}$ denotes the side limits (13), c_0 the value at $b=b_s$ [Eq. (14)], color indicating the antiparallel (c_-, c_0) or parallel (c_+) type. The dotted line depicts the magnetization jump (15a) and (15b). These curves hold for any spin pair and Hamiltonian of the form (1) (attractive case).

attained by the concurrence in fully symmetric states [10]). As θ increases, C_- decreases while C_+ becomes maximum at $\theta \approx 1.6/\sqrt{n}$ [see Eq. (13)], vanishing both for $\theta \to \pi/2$ ($\chi \to 0$) if $n>2$.

In the attractive case the values (11b) represent the *universal side limits* $C_{\pm} = \lim_{b\to b_s^{\pm}} C_l(b)$ of the GS concurrences $C_l(b)$ at $b=b_s$, valid for *any* separation l or interaction range. For $\chi \to 1$ they correctly approach those for the $|1\rangle \to |0\rangle$ transition taking place at $b=b_c$ in the *XXZ* limit [11] (where $b_s \to b_c$).

The concurrence jump $C_- - C_+$ determines, noticeably, the concurrence $C_0 \equiv C_l(\rho_0)$ in the GS mixture (9),

$$C_0 = \frac{1}{2}(C_- - C_+) = (1-\chi)\frac{\chi^{n-1}}{1-\chi^n} \qquad (12)$$

[see also Eq. (17)], which is of antiparallel type. It is a decreasing function of θ, starting at $1/n$ for $\theta \to 0$. In the attractive case, Eq. (12) represents the common $T\to 0$ limit of the thermal concurrences $C_l(T)$ at $b=b_s$ for any separation l and coupling range.

Although for fixed $\chi<1$, $C_{\pm}$ become exponentially small as n increases, the rescaled concurrences $nC_{\pm}$ *remain finite for small anisotropy* $\chi = 1-\delta/n$. For large n and fixed δ, we obtain from Eqs. (11) and (12) the n-independent limits

$$c_{\pm} \equiv nC_{\pm} \approx \delta e^{-\delta/2}/(1 \pm e^{-\delta/2}), \qquad (13)$$

$$c_0 \equiv nC_0 \approx \delta e^{-\delta}/(1-e^{-\delta}), \qquad (14)$$

depicted in Fig. 1. While c_- and $c_0 = c_-(2\delta)/2$ are decreasing functions of δ, c_+ is maximum at $\delta = 2[1+w(e^{-1})] \approx 2.56$, where $c_+ = 2w(e^{-1}) \approx 0.56$ [$w(x)$ is the product log function, such that $x = we^w$]. We note also that $c_0 > c_+$ for $\delta < 2\ln 2$.

The mean rescaled concurrence $(c_+ + c_-)/2$ determines, remarkably, the *total magnetization jump* at $b=b_s$:

$$\Delta M \equiv \langle\theta_-|S_z|\theta_-\rangle - \langle\theta_+|S_z|\theta_+\rangle$$

$$= n\sin^2\theta \frac{\cos^{n-1}\theta}{1-\cos^{2n}\theta} = \frac{1}{2}(c_+ + c_-)\sqrt{\chi}, \qquad (15a)$$

which represents as well the slope of the *energy gap* $\Delta E \approx (b-b_s)\Delta M$ between the odd and even GS at $b=b_s$. For large n and fixed δ,

$$\Delta M \approx (c_+ + c_-)/2 = \delta e^{-\delta/2}/(1-e^{-\delta}), \qquad (15b)$$

remaining finite and providing a direct way to determine the average rescaled concurrence at b_s.

As an illustration, we now show exact results for the concurrence in (I) a fully connected chain with constant r_l [12] and (II) a chain with nearest-neighbor coupling ($r_l = \delta_{l,1} + \delta_{l,n-1}$). In (I) we set $r_l = 2/(n-1)\,\forall\, l$ such that $r=1$ in (I) and (II) [Eq. (5)]. The factorizing field (6) and the energy (7) are then the same in (I) and (II) for fixed $v_{x,y,z}$. We will consider $v_x > 0$ and $v_z = 0$ (*XY* case).

In (I), the GS can be obtained numerically by diagonalizing H in the subspace of maximum total spin states (to which it belongs) as $[H,S^2]=0$:

$$H_I = bS_z - \sum_{\mu=x,y} v_\mu\left(S_\mu^2 - \tfrac{1}{4}n\right)/(n-1).$$

The fixed parity GS is then of the form $\sum_{k\,\text{even (odd)}} w_k S_+^k|0\rangle$, leading to l independent elements $\alpha_l^+ = \langle S_+^2\rangle/c_n$, $\alpha_l^- = (n^2/4 - \langle S_z^2\rangle)/c_n$, $\langle s_z^i s_z^{i+l}\rangle = (\langle S_z^2\rangle - n/4)/c_n$, with $c_n = n(n-1)$.

In (II), the Hamiltonian can be solved *analytically* for any finite n by means of the Jordan-Wigner transformation [13], which allows one to rewrite H, *for each value* ($\pm$) of the parity P_z, as a quadratic form in fermion operators $c_i^\dagger$, c_i defined by $c_i^\dagger = s_i^+ \exp[-i\pi\sum_{j=1}^{i-1} s_j^+ s_j^-]$:

$$H_{II}^{\pm} = \sum_{i=1}^{n} b\left(c_i^\dagger c_i - \frac{1}{2}\right) - \frac{1}{2}\eta_i^{\pm}(v_+ c_i^\dagger c_{i+1} + v_- c_i^\dagger c_{i+1}^\dagger + \text{H.c.})$$

$$= \sum_{k\in K_{\pm}} \lambda_k\left(a_k^\dagger a_k - \frac{1}{2}\right), \qquad (16)$$

where $n+1 \equiv 1$, $\eta_i^- = 1$, $\eta_i^+ = 1 - 2\delta_{in}$, and

$$\lambda_k^2 = (b - v_+\cos\omega_k)^2 + v_-^2\sin^2\omega_k, \quad \omega_k = 2\pi k/n,$$

with $K_+ = \{\frac{1}{2},\ldots,n-\frac{1}{2}\}$, $K_- = \{0,\ldots,n-1\}$ [i.e., k half-integer (integer) for positive (negative) parity]. The diagonal form (16) is obtained through a discrete *parity-dependent* Fourier transform [11] $c_j^\dagger = \frac{e^{i\pi/4}}{\sqrt{n}}\sum_{k\in K_\pm} e^{-i\omega_k j} c_k'^\dagger$, followed by a BCS transformation $c_k'^\dagger = u_k a_k^\dagger + v_k a_{n-k}$, $c_{n-k}' = u_k a_{n-k} - v_k a_k^\dagger$ to quasiparticle fermion operators a_k, $a_k^\dagger$, with $u_k^2, v_k^2 = \frac{1}{2}[1 \pm (b - v_+\cos\omega_k)/\lambda_k]$. For $b \geq 0$ we set $\lambda_k \geq 0$ for $k \neq 0$ and $\lambda_0 = v_+ - b$, such that the quasiparticle vacuum in H_{II}^- is odd and the lowest energies for each parity are $E_{II}^{\pm} = -\frac{1}{2}\sum_{k\in k_\pm}\lambda_k$. At $b = b_s = \sqrt{v_x v_y}$ [Eq. (6)], $\lambda_k = v_+ - b_s\cos\omega_k$ and $E_{II}^+ = E_{II}^- = -nv_+/2$, in full agreement with Eq. (7).

The concurrences in the fixed parity GS can be obtained from the contractions $f_l \equiv \langle c_i^\dagger c_j\rangle_{\pm} - \frac{1}{2}\delta_{ij}$, $g_l \equiv \langle c_i^\dagger c_j^\dagger\rangle_{\pm}$ and the use of Wick's theorem [13], leading to $\langle s_z^i\rangle = f_0$, $\langle s_z^i s_z^j\rangle = f_0^2 - f_l^2 + g_l^2$, and $\alpha_l^{\pm} = \frac{1}{4}[\det(A_l^+) \mp \det(A_l^-)]$, with $(A_l^{\pm})_{ij} = 2(f_{i-j\pm 1} + g_{i-j\pm 1})$ $l\times l$ matrices.

In both (I) and (II), as b increases from 0^+, $[n/2]$ GS parity transitions $\pm \to \mp$ take place if $\chi \in (0,1]$ (as in the *XXZ* case [11]), the last one $(-\to +)$ at $b=b_s$. They are clearly visible for low n if χ is not small, i.e., if $\delta = n(1-\chi)$ is not too large, as seen in Fig. 2 for $n=10$ qubits. For $b \to b_s^{\pm}$, all GS concurrences $C_l^{\pm}$ are seen to approach the same side limits (11b) in both (I) and (II), which are non-

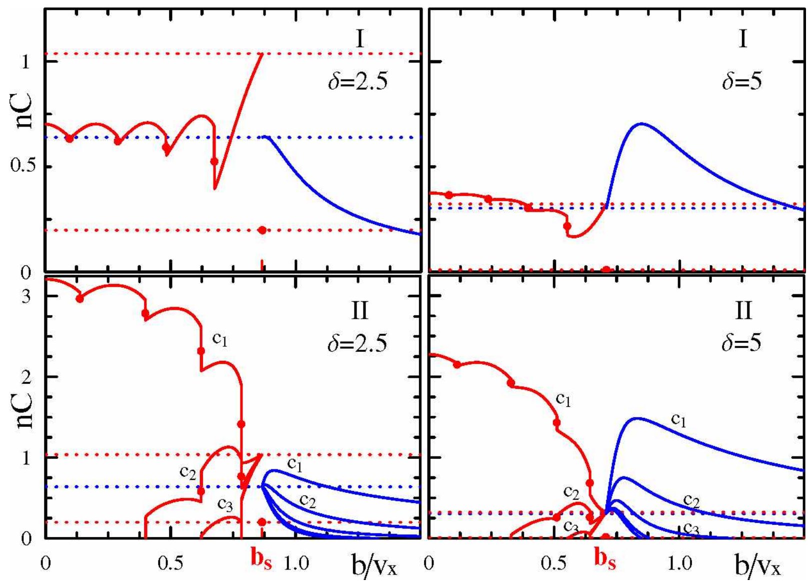

FIG. 2. (Color online) Ground state rescaled concurrences c_l vs magnetic field b for $n=10$ qubits and anisotropy $\chi=1-\delta/n=0.75$ (left) and 0.5 (right). Top panels correspond to a fully connected chain [case (I)], where c_l is the same for all separations l, bottom panels to nearest-neighbor coupling [case (II)], where c_l is shown for all separations. Horizontal dotted lines indicate the limit values (11b) and (12). At $b=b_s$, c_l changes from antiparallel (red) to parallel (blue) type, the side limits being nonzero and identical $\forall l$, and the same in (I) and (II). Large dots indicate concurrence values at the $n/2$ parity transitions (for the mixture of both ground states), with that at $b=b_s$ given by Eq. (12) $\forall l$ in (I) and (II).

negligible. For $\delta=2.5$, $C_l^{\pm}$ reaches in fact its *maximum* for $b\to b_s\approx 0.87v_x$ in (I), and also in (II) if $l>2$. For $\delta=5$ the side limits are still noticeable but nearly coincident, implying a negligible c_0 [Eq. (12)].

The behavior of $c_l^{\pm}$ for $n=50$ qubits at the same values of δ (now $\chi=0.95$ and 0.9), depicted in Fig. 3, is seen to be the same as in Fig. 2 in the vicinity of b_s. All pairs become entangled as $b\to b_s$, with C_l approaching the common values (11) in (I) and (II), now well-approximated by Eqs. (13).

Within each parity subspace, the factorizing field is distinguished as that where *all* GS concurrences $C_l^{\pm}$ *cross* at the values (11) (rather than vanish), as seen in the top panel of Fig. 4. Moreover, in (II) the *ordering* of concurrences $C_l^{\pm}$ becomes *inverted* at $b=b_s$: C_l^- (C_l^+) *increases with increasing separation* l for b just above (below) b_s, as $C_l^{\pm}(b)$ is linear close to b_s. Note that $C_l^+(b)$ vanishes at a lower field $b_l^+<b_s$, becoming *antiparallel* for $b<b_l^+$.

On the other hand, at sufficiently low temperatures, b_s can be identified as the field where all thermal concurrences $C_l(T)$ *cross* at the value (12), as seen in the bottom panel of Fig. 4. $C_l(T)$ vanishes at a slightly *larger* l-dependent field $b_l(T)>b_s$, remaining *antiparallel* until $b_l(T)$. To understand this effect, we note that in a general mixture

$$\rho_q = q|\theta_+\rangle\langle\theta_+| + (1-q)|\theta_-\rangle\langle\theta_-|, \quad q\in[0,1]$$

the concurrence $C(q)\equiv C_l(\rho_q)$ is

$$C(q) = |1-q/q_c|C_-, \quad q_c = \frac{1}{2}(1+\cos^n\theta), \tag{17}$$

which generalizes Eq. (12) (recovered for $q=1/2$). $C(q)$ is antiparallel (parallel) for $q<q_c$ ($>q_c$) *and zero at* $q=q_c>1/2$, where ρ_q becomes completely separable [$\rho_{q_c}=\frac{1}{2}(|\theta\rangle\langle\theta|+|-\theta\rangle\langle-\theta|)$]. Separability requires then a slightly *greater* weight in $|\theta_+\rangle$ due to its lower concurrence. Hence at low $T>0$, $C_l(T)$ vanishes and changes from antiparallel to

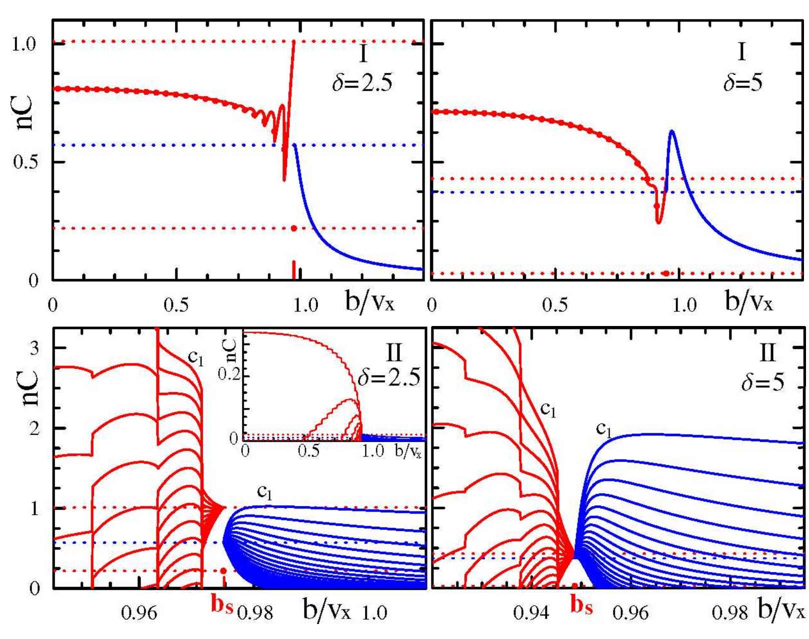

FIG. 3. (Color online) Same details as Fig. 2 for $n=50$ qubits. Bottom panels depict results in the vicinity of b_s, the inset those in the full interval. The limit values at $b=b_s$ are again the same for all separations and identical in (I) and (II).

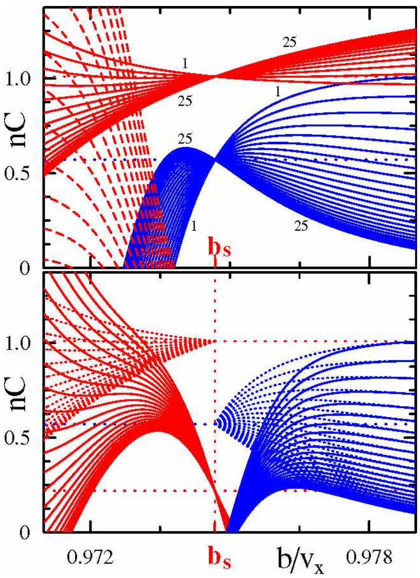

FIG. 4. (Color online) Top: Rescaled concurrences c_l in the lowest state for odd (upper curves, in red) and even (lower curves, in blue) parity P_z, for case (II) with $\delta=2.5$ and $n=50$. Numbers indicate the separation l. At $b=b_s$ an inversion in the ordering of the c_l's with l takes place. Dashed lines depict the antiparallel concurrence in the even states. Bottom: Thermal concurrences in the previous system at $kT=5\times10^{-4}v_x$ (solid lines) and at $T=0$ (dotted lines).

parallel at a slightly *larger* field $b_l(T)>b_s$, where the positive parity GS has a higher weight in the thermal mixture. In case (II) this entails the surprising result that in the narrow interval $b_s<b<b_1(T)$, the thermal concurrence $C_l(T)$ *will increase with increasing* l, since it is still driven by the lowest odd state $\forall l$.

Let us finally mention that for any chain partition $(L,n-L)$, the Schmidt number of the states (8a) is 2. Their Schmidt decomposition [1] is

$$|\theta_\pm\rangle = \sqrt{p^+_{L\pm}}|\theta^L_+\rangle|\theta^{n-L}_\pm\rangle + \sqrt{p^-_{L\pm}}|\theta^L_-\rangle|\theta^{n-L}_\mp\rangle, \qquad (18)$$

$$p^\nu_{L\pm} = \frac{(1+\nu\cos^L\theta)(1\pm\nu\cos^{n-L}\theta)}{2(1\pm\cos^n\theta)} \quad (\nu=\pm), \qquad (19)$$

where $|\theta^L_\pm\rangle=(|\theta^L\rangle\pm|-\theta^L\rangle)/\sqrt{2(1\pm\cos^L\theta)}$ denotes the analogous fixed parity states for L spins and $p^\nu_{L\pm}$ the eigenvalues of the ensuing reduced density. The entanglement between L and $n-L$ spins can be measured through the entropy $S^\pm_L=-\Sigma_\nu p^\nu_{L\pm}\log_2 p^\nu_{L\pm}$ or equivalently, the "global" concurrence $C^\pm_L=\sqrt{2[1-\Sigma_\nu(p^\nu_{L\pm})^2]}$ [14] (square root of the tangle for a pure state), which is just an increasing function of $S^\pm_L$ ($S^\pm_L,C^\pm_L\in[0,1]$):

$$C^\pm_L = 2\sqrt{p^+_{L\pm}p^-_{L\pm}} = \frac{\sqrt{(1-\chi^L)(1-\chi^{n-L})}}{1\pm\chi^{n/2}},$$

where we have replaced $\cos\theta=\sqrt{\chi}$. Hence $C^-_L\geq C^+_L$, with $C^\pm_L$ *increasing* functions of θ, i.e., decreasing functions of χ, *in contrast* with the pairwise concurrences $C_\pm$. For $\theta\to\pi/2$ $(\chi\to0)$, $C^\pm_L\to1$ (GHZ limit), whereas for $\theta\to0$ $(\chi\to1)$, $C^+_L\to0$ but

$$C^-_L\to2\sqrt{L(n-L)}/n,$$

(W-state limit), in which case $S^-_L\approx(L/n)[1-\log_2(L/n)]$ for $L\ll n$. Thus within the bounds imposed by a Schmidt number 2, the behavior of $S^\pm_L$ and $C^\pm_L$ with L is "noncritical" (i.e., saturated) for low χ (large δ) and "critical" (nonsaturated) for $\chi\to1$ (low δ) and negative parity. It is also verified that $C^\pm_1\geq\sqrt{n-1}C_\pm$ (in agreement with the general inequality $C^2_{L=1}\geq\Sigma^{n-1}_{l=1}C^2_l$ [15]), with saturation reached for $\theta\to0$, where $C^\pm_1/C_\pm\approx\sqrt{n-1}[1+\frac{1}{4}\theta^2(n-2)]$.

In summary, we have shown that due to the S_z parity conservation, the GS of finite cyclic chains with attractive couplings of the form (1) remains entangled as the factorizing field b_s is approached, undergoing at b_s the last parity transition and exhibiting for $b\to b^\pm_s$ universal entanglement properties, "intermediate" between those of GHZ and W states. This field plays thus the role of a "quantum critical field" for small chains, with the pairwise concurrence reaching infinite range and approaching distinct side limits which are independent of the pair separation and interaction range. Their average is directly measurable through the GS magnetization jump (15a) and (15b), which provides then a signature of the present effects, while their difference determines the concurrence of the GS mixture (12). These effects remain appreciable for increasing n if the anisotropy becomes sufficiently small (finite δ), i.e., for chains close to the XXZ limit. Moreover, within a fixed parity subspace (and also at sufficiently low $T>0$), b_s is singled out as the field where all pairwise concurrences cross, the ordering with separation becoming inverted as b crosses b_s. Type, range, and even ordering of the pairwise entanglement can thus be controlled by tuning the field around b_s.

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