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A necessary condition for EPT graphs and a new family of minimal forbidden subgraphs

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Abstract

An undirected graph G is called an EPT graph if it is the edge intersection graph of a family of paths in a tree. In this paper we define the concept of satellite of a clique and we give a necessary condition to be an EPT graph based on satellites of cliques. We characterize the minimal graphs which do not satisfy the previous condition, as a consequence we present a finite family of minimal forbidden subgraphs for the EPT class.

1 Introduction

The intersection graph of a set family is a graph whose vertices are the members of the family and the adjacency is defined by a non-empty intersection of the corresponding members. Some classes of graphs defined as intersection are hereditary and can be characterized by minimal forbidden induced subgraphs. Classical examples are interval graphs and chordal graphs.

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A **chordal graph** is a graph without induced cycles of length at least four. Gravril [1] proved that a graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree, considering vertex intersection.

An **interval graph** is the intersection graph of a family of closed intervals on the real line, or equivalently the intersection graph of a family of subpaths of a path. This class was characterized by forbidden subgraphs by Lekkerkerker and Boland [5].

Special classes of graphs are defined imposing restrictions on trees, subtrees and considering intersection on vertices or edges of subtrees, and many of them are hereditary.

Let P be a family of paths on a host tree T. Two types of intersection graphs from the pair $\langle P,T\rangle$ are defined, namely VPT and EPT graphs. The **edge intersection graph of P**, $EPT(P)$, has vertices which correspond to the members of P, and two vertices are adjacent in EPT(P) if and only if the corresponding paths in P share at least one edge in T. An undirected graph G is called an **edge intersection graph of paths in a tree (EPT)** if $G = EPT(P)$ for some P and T, and $\langle P,T\rangle$ is called an EPT representation of G. Similarly, the **vertex intersection graph of P**, $VPT(P)$, has vertices which correspond to the members of P, and two vertices are adjacent in $VPT(P)$ if and only if the corresponding paths in P share at least one vertex in T. An undirected graph G is called a **vertex intersection graph of paths in a tree (VPT)** if $G = VPT(P)$ for some P and T, and $\langle P,T\rangle$ is called a VPT representation of G.

VPT and EPT graphs are incomparable families of graphs. However, when the maximum degree of the host tree is restricted to 3 the family of VPT graphs coincides with the family of EPT graphs.

The complexity of recognizing VPT graphs is polynomial, but the recognition of EPT graphs is an NP-complete problem [3].

Recently VPT graphs, also called path graphs, were characterized by minimal forbidden induced subgraphs [6, 8]. The present paper shed more light about this problem in EPT graphs, it is structured as follows: In Section 2 we speak about cliques of an EPT graph. In Section 3 we give our theorem that show a necessary condition to be an EPT graph. In Section 4 we present a finite family of minimal forbidden subgraphs for the EPT class. Finally, in Section 5 we show an open question.

2 Basic results about cliques in EPT graphs

In EPT graphs the cliques have a particular form which is very important for our result [3]. A **clique** of a graph G is a maximal complete set of G, that is a maximal subset of vertices of G pairwise adjacent.

The **claw graph** $K_{1,3}$ consists of one central vertex and three edges incident on it. These edges are called the legs of the claw.

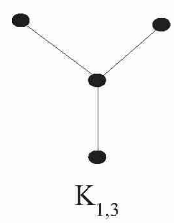

Figure 1: Claw graph

Let $\langle P, T\rangle$ be an EPT representation of G. For any edge e of T, let $\mathsf{P}[e] = \{\mathsf{P} \in P/e$ is an edge of $\mathsf{P}\}$. For any claw $K_{1,3}$ in T, let $\mathsf{P}[K_{1,3}]=$ $\{\mathsf{P} \in P/\mathsf{P}$ contains two legs of $K_{1,3}\}$. The collection $\mathsf{P}[e]$ corresponds to a clique in G and is called an **edge clique**. Similarly, $\mathsf{P}[K_{1,3}]$ also corresponds to a clique in G and is called a **claw clique**.

Theorem 2.1. *[3] Let $\langle P, T\rangle$ be an EPT representation of G. Any clique of G corresponds to either a subcollection of paths of the form $\mathsf{P}[e]$ for some edge e in T or of the form $\mathsf{P}[K_{1,3}]$ for some claw $K_{1,3}$ in T.*

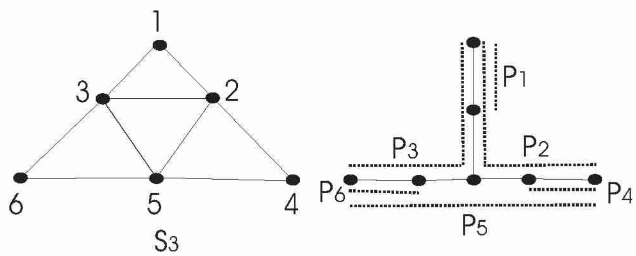

Figure 2: An EPT representation of the sun S_3. The central triangle $\{2, 3, 5\}$ is a claw clique; the other three triangles are edge cliques.

3 A necessary condition for EPT graphs

In this section we give a necessary condition to be an EPT graph based on the following concepts:

Let $G = (V, E)$ and let $v \in V$. The **neighborhood of v in G** is the set $N(v) = \{w \in V, w \neq v \ / vw \in E\}$.

Definition 3.1. *Let C be a clique of G. A vertex v of G is a **satellite of C** if $v \notin C$ and $B_v = N(v) \cap C$ is a non-empty proper subset of C.*

*The set B_v is called the **base of v** and it is said **minimal** if no other base of a satellite of C is properly contained in B_v.*

Theorem 3.1. *Let C be a clique of an EPT graph G. If $w \in C$ then w belongs to at most two different minimal bases of non-adjacent satellites of C.*

Proof: Let $\langle P, T \rangle$ be an EPT representation of G. By Theorem 2.1 we know that C is an edge clique or a claw clique. We consider the two possibilities:

Case (1): $C = \{v_1, ..., v_k\}$ is an edge clique, that is each path P_{v_i}, with $1 \leq i \leq k$, representing vertices of C in $\langle P, T \rangle$, have an edge $e = xy$ of T in common.

Let $w \in C$, suppose on the contrary, that w is in three different minimal bases of non-adjacent satellites of C, say y_1, y_2, y_3. Let P_w be the corresponding path representing the vertex w in $\langle P, T \rangle$, then $|V(P_w) \cap V(P_{y_i})| \geq$

2, for $i = 1, 2, 3$. Since P_w is a path that goes through edge e and w is adjacent to y_i, for $i = 1, 2, 3$, we have that at least two of the paths P_{y_i}, $1 \leq i \leq 3$, must be in a same connected component of the graph $(T - e)$. Suppose that P_{y_1} and P_{y_2} are those paths. Since y_1 and y_2 are both adjacent to w we have that the paths P_{y_1} and P_{y_2} must have at least one edge in common with the path P_w, and this edge can not be the same for both paths since y_1 is non-adjacent to y_2. Let e_1, e_2 be these edges. It is clear that $e_1 \neq e$ and $e_2 \neq e$, because $y_1 \notin C$ and $y_2 \notin C$, by the definition of satellite of C. Then, since P_w is a path these edges e_1, e_2 will be in a same path P of $(T - e)$. Suppose that e_1 is closer to e than e_2 in this path P, then each path P_{v_i}, with $v_i \in C$, that pass through e_2 will pass through e_1 too, that is, each path P_{v_i}, with $v_i \in C$, that intersect P_{y_2} in at least one edge, will intersect P_{y_1} in at least one edge too, which contradicts the minimality of the bases.

Case (2): $C = \{v_1, ..., v_k\}$ is a claw clique in which the claw is formed by the edges e_1, e_2, e_3 and a central vertex z. Then every path P_{v_i}, $1 \leq i \leq k$, has exactly two of the edges e_i, $i = 1, 2, 3$, since they are paths which touch each other in at least two vertices of the host tree T. Let $w \in C$, suppose, on the contrary, that w is in three different minimal bases of non-adjacent satellites of C, say y_1, y_2, y_3. Let P_w be the path representing w in $\langle P, T \rangle$. Observe that $(T - z)$ is a graph which has three connected components, say C_1, C_2 and C_3. If the paths P_{y_i} corresponding to the satellites y_i, $1 \leq i \leq 3$, intersect more than one connected component of $(T - \{\text{z}\})$ we are in the case (1). Then each P_{y_i}, $1 \leq i \leq 3$, intersects exactly one connected component of $(T - \{\text{z}\})$. Suppose that P_{y_i} intersects C_i, for $i = 1, 2, 3$. Since w is adjacent to y_i, $1 \leq i \leq 3$, we have that the path P_w must have at least one edge in common with each path P_{y_i}, $1 \leq i \leq 3$. But since the paths P_{y_i} intersect only the connected component C_i, for each $1 \leq i \leq 3$, P_w would have e_1, e_2 and e_3 as edges, which contradicts the fact that P_w is a path. $\square$

4 Some new minimal forbidden subgraphs for the EPT class

In this section it will be analyzed if new minimal forbidden graphs for EPT class could appear from the previous result.

Suppose that C is a clique, y_1, y_2, y_3 are three satellites of C and B_1, B_2, B_3 are bases of y_1, y_2 and y_3, respectively. Since we are interested in minimal configurations which do not satisfy the conditions of the Theorem 3.1 these bases must have the following properties:

1. Exist a unique vertex of C, say 1, such that $1 \in B_i$, for $i = 1, 2, 3$,
2. $B_i \nsubseteq B_j$, for $i \neq j$, $i = 1, 2, 3$, $j = 1, 2, 3$,
3. $\bigcup B_i = C$,
4. $B_i \subsetneq C$, for $i = 1, 2, 3$.

Case (1): Suppose that the vertices of C, different of 1, belong to at most one base. Note that there must be exactly one vertex, different of 1, belonging to each base, because of 2. and the minimality. Suppose that $2 \in B_1$, $3 \in B_2$ and $4 \in B_3$ (See Figure 3).

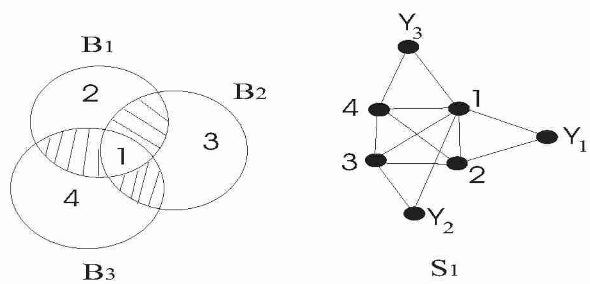

Figure 3: The vertices of C, different of 1, belong to at most one base, and the graph obtained in this case.

Case (2): Suppose that exactly one vertex of C, different of 1, belongs to two different minimal bases, say $2 \in B_1 \cap B_2$. Since the bases are minimal and different there must be at least one vertex of C belonging only to B_2, say $3 \in B_2$. Now, there must exist a vertex of C belonging only to B_1, because in other case we would have that $B_1 \subsetneq B_2$, suppose that $4 \in B_1$. For the same reason, there must exit a vertex of C belonging only to B_3, say $5 \in B_3$, in other case we would have that $B_3 \subsetneq B_1$ or $B_3 \subsetneq B_2$. But note that in this case if we remove the vertex 2 we return to the previous case. Then the graph obtained has S_1 as an induced subgraph (See Figure 4).

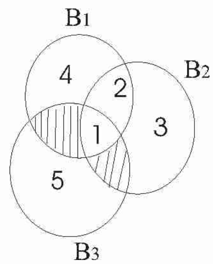

Figure 4: Exactly one vertex of C, different of 1, belongs to two different minimal bases.

Case (3): Suppose that exactly two vertices of C, different of 1, belong to two different minimal bases, suppose $2 \in B_1 \cap B_2$, $3 \in B_1 \cap B_3$. Now we see that there must be at least one vertex of C belonging only to B_2, because in other case we would have that $B_2 \subsetneq B_1$, suppose that $4 \in B_2$. And there must be at least one vertex of C belonging only to B_3, in other case we would have that $B_3 \subsetneq B_1$, suppose that $5 \in B_3$ (See Figure 5).

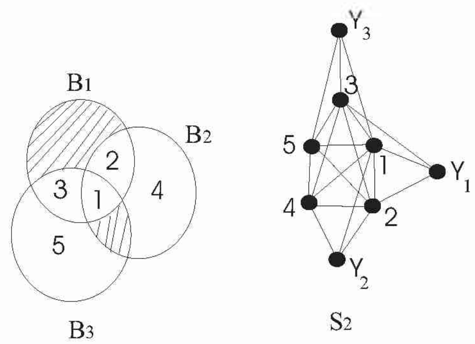

Figure 5: Exactly two vertices of C, different of 1, belong to two different minimal bases, and the graph obtained in this case.

Case (4): Suppose that exactly three vertices of C, different of 1, belong to two different minimal bases, say $2 \in B_1 \cap B_2$, $3 \in B_2 \cap B_3$, $4 \in B_1 \cap B_3$. Note that is not necessary adding any other vertex to the bases because the minimal conditions are made (See Figure 6).

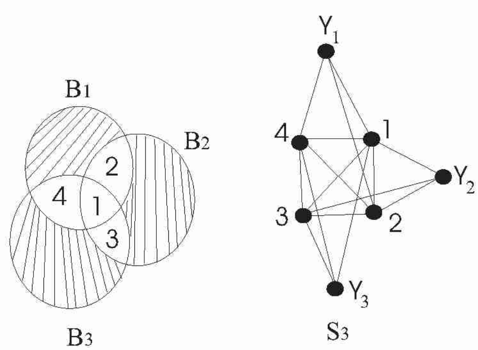

Figure 6: Exactly three vertices of C, different of 1, belong to two different minimal bases, and the graph obtained in this case.

It is easy to see that the subgraphs obtained removing any vertex of S_1, S_2 or S_3 are EPT graphs, that is, every induced subgraph of S_1, S_2 or S_3 is an EPT graph.

Then we have the following theorem:

Theorem 4.1. *S_1, S_2, S_3 are minimal forbidden subgraphs for the EPT class.*

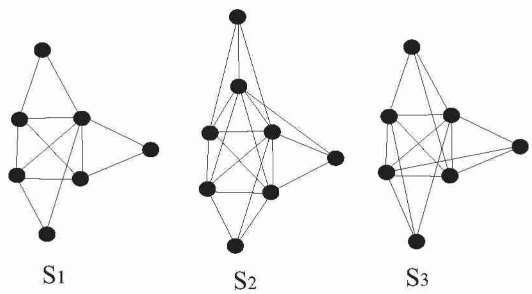

Figure 7: A new family of minimal forbidden subgraphs for EPT graphs.

5 Conclusion

In this work we give a new finite family of minimal forbidden subgraphs for the EPT class, but we know that this family is incomplete because for example A_4 (see Figure 8) is a minimal non-EPT graph [4] and A_4 is not in our family of minimal forbidden subgraphs. Observe that S_1, S_2, S_3 and A_4 are chordal graphs, it could be interesting to know the complete list of chordal graphs which are minimal forbidden subgraphs for the EPT class.

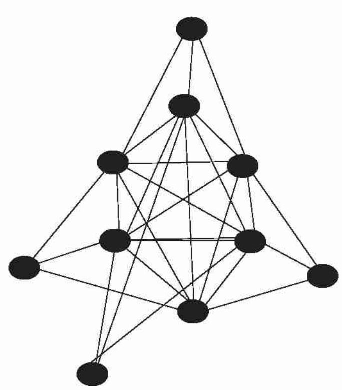

Figure 8: A_4 is not an EPT graph

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