# Notas de Matemática

## The Calderón Projector for an Elliptic Operator in divergence form

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# The Calderón Projector for an Elliptic Operator in divergence form.

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**Abstract**

In this paper we construct the **Calderón Projector** for an elliptic operator in divergence form with Lipschitz coefficients on a $C^1$ domain $\Omega$, and we prove some results about the continuity of this operator on the $L^p(\partial\Omega)$ spaces.

The projection on the Cauchy data, usually called **Calderón Projector**, is one of the most important tools in the study of boundary value problems for elliptic operators.

This projector has been constructed first by A. P. Calderón for elliptic operators with $C^\infty$ coefficients in a $C^\infty$ domain ( in this case, the projector is a pseudodifferential operator), and it allowed one to formulate non local elliptic boundary value problems, thus extending the classic local elliptic boundary value problems (Lopatinsky conditions)[1], see also [11].

For the Laplacian operator, even on a $C^1$ domain, the results of E.Fabes, M.Jodeit and N.Riviere [4] allow us to construct this projector as a continuous operator on $L_1^p(\partial\Omega) \times L^p(\partial\Omega)$, $\forall 1 < p < \infty$ .

In this paper we construct the projector for elliptic operators in divergence form $L = -div(A\nabla)$ on $C^1$ domains when the coefficients of $A$ are Lipschitz functions, and we obtain the same results as for the Laplacian.

In the first section we show how the Calderón Projector is an important tool in the analysis of the solutions of boundary problems like Dirichlet and Neumann, and we recall the classic results about its construction, by means of the layer potentials.

In section 2 we analyze the behavior near the diagonal of the fundamental solutions for the operator $L$, specially the relation with the fundamental solutions for operators in divergence form with constant coefficients.

In sections 3 and 4, we prove some contuinity results on the $L^p(\partial\Omega)$ spaces of the traces single and double layer potentials and the relation with the nontangential limits of the potentials. The main result in this section is Theorem 3.3, where we state that a singular operator with kernel $k(x,y) = \dfrac{\eta(x) - \eta(y)}{|B(x,\phi(x))(x-y,\phi(x)-\phi(y))|^n}$, where

$B$ is an $n \times n$ matrix with Lipschitz coefficients, and $\eta, \phi$ are two Lipschitz function, is a continuous operator on $L^p(\mathbb{R}^n)$.Indeed we prove that it is a Calderón-Zygmund operator (we give the proof in the Appendix).

In section 5 we prove the invertibility results that are necessary for the analysis of the Dirichlet and Neumann problems for $L$ in a $C^1$ domain, then we prove some regularity theorems that are necessary for the construction of the projector.

Finally, in section 6 we construct the Calderón Projector for our case, taking into account the results obtained in the previous sections.

# 1 Preliminaries

In this section, we will show the construction of the **Calderón Projector** in the smooth case.

Let $A(X) = (a_{i,j}(X))_{i,j=1}^n$ be a real, symmetric, $n \times n$ matrix, with $A(X) \in C^\infty(\mathbb{R}^n)$, and uniformly elliptic, i.e. there exists $\lambda > 0$, such that

$$\lambda|\xi|^2 \leq \langle A(X)\xi, \xi\rangle \leq \lambda^{-1}|\xi|^2 \tag{1.1}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Let us now consider the elliptic operator $L = -div(A(X)\nabla)$ in a $C^\infty$ domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. For $X \in \Omega$ and $Q \in \partial\Omega$, the potential operators of single layer $\mathcal{S}$, and double layer $\mathcal{T}$ are defined by

$$\mathcal{S}f(X) = \int_{\partial\Omega} K(X,Q)f(Q)dQ, \tag{1.2}$$

$$\mathcal{T}f(X) = \int_{\partial\Omega} \partial\eta_{A(Q)}K(X,Q)f(Q)dQ \tag{1.3}$$

where $K$ is a symmetric fundamental solution for $L$, $\partial\eta_{A(Q)} = \langle A(Q)\nabla_Q, N_Q\rangle$, $N_Q$ is the unit normal vector to $\partial\Omega$ in $Q$, and we have denoted by $dQ$ the surface measure.

Take $\Omega' \supset \Omega$ and $u(X) \subset C_0^\infty(\Omega')$ to be a solution for $Lu = 0$ in $\Omega$, then we have the well known representation formula:

$$u(X) = \int_{\partial\Omega} -\partial\eta_{A(Q)}K(X,Q)u(Q) + K(X,Q)\partial\eta_A u(Q)dQ \tag{1.4}$$

By means of the layer potential operators, we can write

$$u(X) = -\mathcal{T}u(X) + \mathcal{S}(\partial\eta_A u)(X), \quad X \in \Omega \tag{1.5}$$

And taking limits (non-tangential) for $X \to P \in \partial\Omega$, we have

$$u(P) = \frac{1}{2}u(P) - Tu(P) + S(\partial\eta_A u)(P) \tag{1.6}$$

where $S$ and $T$ are the respectives traces of the single and double layer potentials, i.e

$$Tf(P) \quad = vp \int_{\partial\Omega} \langle A(Q)\nabla_Q K(P,Q), N_Q \rangle f(Q)dQ. \tag{1.7}$$

and since the singularity of $K(X,Q)$ is integrable, even when $X \in \partial\Omega$, the corresponding trace is given by

$$Sf(P) = \int_{\partial\Omega} K(P,Q)f(Q)dQ, \ P \in \partial\Omega. \tag{1.8}$$

If we consider for example, the Dirichlet problem for $L$ in $\Omega$, with boundary value $u = f$ in $\partial\Omega$, we obtain the following identity:

$$Sg = (T + \frac{1}{2}I)f \tag{1.9}$$

where $g = S^{-1}(T + \frac{1}{2}I)f = \partial\eta_A u$ is the unknown data on $\partial\Omega$.

We can also see the problen in a diferent way: we define the operators

$$\mathcal{D}f(X) = \partial\eta_{A(X)} \int_{\partial\Omega} \partial\eta_{A(Q)} K(X,Q)f(Q)dQ \tag{1.10}$$

and

$$Df(P) = \lim_{X \to P} \partial\eta_{A(P)} \mathcal{T}f(X), \ P \in \partial\Omega. \tag{1.11}$$

Now, taking the conormal derivative on $\partial\Omega$ of equation (1.5), and considering the boundary values $f$ and $g$, we have:

$$Df = (T^t - \frac{1}{2}I)g \tag{1.12}$$

where $T^t$ is the adjoint operator of $T$.

On the other hand, given two functions, $f$ and $g \in C^\infty(\partial\Omega)$, we define

$$\mathcal{U}(f,g)(X) = -\mathcal{T}f(X) + \mathcal{S}g(X). \tag{1.13}$$

It is obviously a solution for $Lu = 0$ in $\Omega$.And

$$\lim_{X \to P \in \partial\Omega} \mathcal{U}(f,g)(X) = (\frac{1}{2}I - T)f(P) + Sg(P) \tag{1.14}$$

(taking this limit by nontangential inner cones to $\partial\Omega$).

Also under enough regularity hypotheses,

$$\lim_{X \to P \in \partial\Omega} \partial\eta_A \mathcal{U}(f,g)(X) = -Df(P) + (\frac{1}{2} + T^t)g(P). \tag{1.15}$$

Now, if we write the following matrix operator:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2}I - T & S \\ -D & \frac{1}{2}I + T^t \end{pmatrix} \tag{1.16}$$

we have that $ImP \subset \{(u|_{\partial\Omega}, \partial\eta_A u|_{\partial\Omega}) : Lu = 0 \text{ en } \Omega\}$. As we have just seen, if we take $(f,g) \in \{(u|_{\partial\Omega}, \partial\eta_A u|_{\partial\Omega}) : Lu = 0 \text{ in } \Omega\}$, then $P\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$.

**Definition 1.1** *With the hipotheses and the notation considered above, the* **Calderón Projector** *for $L$ in $\Omega$ is the operator* **P** *given by 1.16. And this operator has the following properties:*

- $\mathbf{P} : C^\infty(\partial\Omega) \times C^\infty(\partial\Omega) \to C^\infty(\partial\Omega) \times C^\infty(\partial\Omega)$
- $\mathbf{P}^2 = \mathbf{P}$
- $Im\mathbf{P} = \{(u|_{\partial\Omega}, \partial\eta_A u|_{\partial\Omega}) : Lu = 0 \text{ } in \text{ } \Omega\}$
- **P** *is a pseudodifferential operator.*

For the details of the construction of this operator in the smooth case (as we have just considered, and for more general elliptic operators) see for instance, Calderón [1].

As we have seen, the projector **P** depends on the fundamental solutions of $L$ (really we have a family of projectors!), then we will construct **P** for a suitable fundamental solution of the operator $L = -div(A\nabla)$, when the matrix $A$ has Lipschitz coefficients and $L$ is defined on a $C^1$ domain.

## 2 About the singularity of fundamental solutions

The main tool in this section is the work of Grüter and Widman [5] where they analyze the properties of the Green function for an elliptic operator, with even less regularity than Lipschitz coefficients.

Let $A(X) = (a_{i,j}(X))_{i,j=1}^n$ be a real, symmetric, $n \times n$ matrix, uniformly elliptic, with Lipschitz coefficients, i.e., $|a_{i,j}(X) - a_{i,j}(Y)| \leq c_{i,j}|X - Y| \ \forall\ X, Y \in \mathbb{R}^n$. We denote $A' = \max c_{i,j}, 1 \leq i \leq n, 1 \leq j \leq n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded and Lipschitz domain and consider the elliptic operator $L = -div(A(X)\nabla)$ on this domain.

**Definition 2.1** *We say that $u \in W^2_{1,loc}(\Omega)$ (where $W^2_1(\Omega) = \{f \in L^2(\Omega) : \int_\Omega |f|^2 + \int_\Omega |\nabla f|^2 < \infty\}$, the usual Sobolev space) is a solution for $Lu = 0$ in $\Omega$ if*

$$\int_\Omega u(Y)L(\phi(Y))dY = 0, \ \forall\ \phi \in C_0^\infty(\Omega)$$

**Definition 2.2** *We say that $K(X, Y)$ is a fundamental solution for $L$ in $\Omega$ if "$L(K(X,.)) = LK(.,X)) = \delta_X$", and if we put $u(X,Y) = K(X,Y) - G(X,Y)$, then $u \in W^2_1(\Omega)$ is a solution for $L$ in the two variables and $G(X,Y)$ is the Green function for $L$ in $\Omega$.*

For the definition and properties of the Green function in this case, see for instance [5]. For any fundamental solution, near the diagonal $X = Y$, we have the same behavior as for the Green function and its derivatives proved there, but we need a more accurate analysis of their singularities. Then, we state the following theorem:

**Theorem 2.1** *Let $L$ be the elliptic operator defined above, in a bounded, Lipschitz domain $\Omega \subset \mathbb{R}^n, n \geq 3$, and let $K(X,Y)$ be a fundamental solution for $L$, then $\forall\ X, Y \in \Omega\ :\ X \neq Y$,*

$$|\nabla_Y K(X,Y) - \nabla_Y K^X(X,Y)| \leq C(\Omega, \lambda, n, \delta(X), \delta(Y), A')|X - Y|^{2-n}, \tag{2.1}$$

*where $K^X(Z,Y)$ is a fundamental solution for the elliptic operator with constant coefficients $L^X = -div_Y(A(X)\nabla_Y)$, and $\delta(Z) = dist(Z, \partial\Omega)$.*

$$If\ X, Y \in \tilde{\Omega} \subset\subset \Omega,\ then\ C(\Omega, \lambda, n, \delta(X), \delta(Y)) \leq C(\Omega, \tilde{\Omega}, \lambda, n). \tag{2.2}$$

**Remark 2.1** *As is natural, it is also true that*

$$|K(X,Y) - K^X(X,Y)| \leq C(\Omega, \lambda, n, \delta(X), \delta(Y), A')|X - Y|^{3-n}, \tag{2.3}$$

*but we do not need this estimate. The proof is easily obtained from (2.5) below.*

**Proof.** Let $X \in \Omega$, then in the distribution sense, $\forall\, Y \neq X$, we have

$$L_Y(K(Z,Y) - K^X(Z,Y)) = -div_Y[(A(X) - A(Y))\nabla_Y K^X(Z,Y)] = f_{X,Z}(Y).$$

Moreover $K^X(Z,Y) \in C^\infty(\Omega \setminus \{Y\})$, then we can evaluate this function in $Z = X$, and if we put $u_X(Y) = K(X,Y) - K^X(X,Y)$, we have that

$$Lu_X(Y) = f_X(Y) \; ; \text{ with } |f_X(Y)| \leq C_{n,A'}|X - Y|^{1-n}. \tag{2.4}$$

Then, for $u_X(Y)$ we have the following representation formula:

$$u_X(Y) = -\int_\Omega f_X(Z)G(Z,Y)dZ + \int_{\partial\Omega} \langle A(Q)\nabla_Q G(Y,Q), N_Q\rangle\, u_X(Q)dQ \tag{2.5}$$

$\forall\, X, Y \in \Omega : X \neq Y$. (We don't prove this formula here, it follows taking into account that A has Lipschitz coefficients and the estimates in [5]). Now, differentiating with respect to $Y_i$, we see

$$\partial_{Y_i} u_X(Y) = -\int_\Omega f_X(Z)\partial_{Y_i} G(Z,Y)dZ + \int_{\partial\Omega} \langle A(Q)\partial_{Y_i}\nabla_Q G(Q,Y), N_Q\rangle\, u_X(Q)dQ.$$

then

$$\begin{aligned} |\nabla_Y u_X(Y)| &\leq C_{\lambda,n,A'} \int_\Omega |Z - X|^{1-n}|Z - Y|^{1-n}dZ + C_\lambda \int_{\partial\Omega} |Q - Y|^{-n}|Q - X|^{2-n}dQ \\ &\leq C_{\lambda,n,A'}|X - Y|^{2-n} + C_\lambda \delta(Y)^{-n}\delta(X)^{2-n}|\partial\Omega| \end{aligned}$$

Then for each $X \in \Omega$, and $\forall\, Y \in \Omega$

$$|\nabla_Y K(X,Y) - \nabla_Y K^X(X,Y)| \;\leq\; C(\Omega, A', \delta(X), \delta(Y), n)|X - Y|^{2-n}.$$

and we also obtain the estimate (2.2) for the function $C(\Omega, A', \delta(X), \delta(Y), n)$ □

## 3 Existence and continuity on $L^p$ of the trace single and double layer potentials for $L$. The relation between the nontangential limits of the double layer potential and its trace.

In the analysis of the layer potentials operators for the Laplacian, we need to know the continuity of operators with kernel $k(x,y) = \dfrac{\eta(x) - \eta(y)}{|(x - y, \phi(x) - \phi(y))|^n}$, with $\eta$ and

$\phi$ Lipschitz functions from $\mathbb{R}^{n-1}$ to $\mathbb{R}$. These operators are included in the family of operators with kernels $k(x,y) = F\left(\frac{\phi(x)-\phi(y)}{(x-y)}\right)\frac{1}{x-y}$, where $F$ is an analytic function in a disc depending on the Lipschistz constant of $\phi$, and it is well known that these are Calderón- Zygmund operators [3]. But in this work, we are interested in operators with kernel $k(x,y) = \frac{\eta(x)-\eta(y)}{|B(x,\phi(x))(x-y,\phi(x)-\phi(y))|^n}$, whith $B(X)$ an (n×n) matrix, uniformly elliptic, and with Lipschitz coefficients. For these operators, in Theorem 3.3, we obtain the same result as in the case $B = I$.

**Definition 3.1** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. It is called a Lipschitz domain, if we can cover a neighborhood of $\partial\Omega$ by finitely many balls $B$ so that, in an appropriate orthonormal coordinate system, $B \cap \Omega = \{(x,s) : s > \phi(x)\} \cap B$, where $(x,s) : x \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$, and $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, with Lipschitz constant $M < \infty$, i.e. $|\phi(x)-\phi(y)| \leq M|x-y|$, $\forall x,y \in \mathbb{R}^{n-1}$, with $\phi(0) = 0$. The domain is called a $C^1$ domain if $\phi$ can be chosen to be of class $C^1$*

Let $K(X,Y)$ be a fundamental solution for $L$ in a domain $\Omega' \subset \mathbb{R}^n, n \geq 3$, and let $\Omega \subset \Omega'$ : $d(\Omega, \partial\Omega') = d > 0$ be a Lipschitz domain.

Now, we recall that the double layer potential in $\Omega$ of $f$ is given by

$$\mathcal{T}f(X) = \int_{\partial\Omega} \langle A(Q)\nabla_Q K(X,Q), N_Q\rangle f(Q)dQ, \quad X \in \Omega \tag{3.1}$$

where $N_Q$ is the outer unit normal to $\partial\Omega$ in $Q$. On the boundary, the trace of the double layer potential is defined as follows: for $P \in \partial\Omega$ let

$$T_\epsilon f(P) = \int_{\partial\Omega, |P-Q|>\epsilon} \langle A(Q)\nabla_Q K(P,Q), N_Q\rangle f(Q)dQ, \tag{3.2}$$

and, when it make sense,

$$Tf(P) = vp \int_{\partial\Omega} \langle A(Q)\nabla_Q K(P,Q), N_Q\rangle f(Q)dQ = \lim_{\epsilon\to 0} T_\epsilon f(P). \tag{3.3}$$

Similarly, the single layer potential of $f$ is given by

$$\mathcal{S}f(X) = \int_{\partial\Omega} K(X,Q)f(Q)dQ, \quad X \in \Omega \tag{3.4}$$

and since the singularity of $K(X,Q)$ is integrable, even when $X \in \partial\Omega$, the corresponding trace is given by

$$Sf(P) = \int_{\partial\Omega} K(P,Q)f(Q)dQ, \quad P \in \partial\Omega. \tag{3.5}$$

Obviously, it is a continuous operator on $L^p(\partial\Omega)$. And, for the trace of the double layer potential, we have the following result:

**Theorem 3.1** *Let $L = -div(A\nabla)$, in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. and $A(X)$ a symmetric $n \times n$ matrix with Lipschitz coefficients such that, $\lambda|\xi|^2 \leq \langle A(X)\xi, \xi\rangle \leq \lambda^{-1}|\xi|^2$, $\forall X \in \overline{\Omega}$.*

*Let $T_\epsilon f(P)$ be the truncated trace double layer potential given by (3.2). Then the mapping $T^* f(P) = \sup_{\epsilon>0} |T_\epsilon f(P)|$ is bounded in $L^2(\partial\Omega)$ ,and $Tf(P) = \lim_{\epsilon\to 0} T_\epsilon f(P)$ exists pointwise a.e and is a Calderón-Zygmund Operator. Moreover, when $\Omega$ is a $C^1$ domain, $Tf(P)$ is compact in $L^p(\partial\Omega)$, $1 < p < \infty$.*

**Proof.** We will not give here all the details of the proof, because most of them are the same than for the Laplacian operator (i.e, when $A = I$), see [4]. We prove the results where there are differences due to the Lipschitz coefficients of the matrix $A$.

First of all, we will use the relation established in Theorem 2.1 for the fundamental solutions: Let us consider the operator $L^X = -div_Y(A(X)\nabla_Y)$ in the domain $\Omega'$. and let

$$K^X(Z,Y) = K^X(Z-Y) = -\frac{1}{(n-2)\omega_n}|detA^{-\frac{1}{2}}(X)||A^{-\frac{1}{2}}(X)(Z-Y)|^{2-n} \quad (3.6)$$

be the homogeneous fundamental solution for $L^X$. Taking now $Z = X$, and differentiating in $Y$, we obtain:

$$\nabla_Y K^X(X,Y) = -\frac{1}{\omega_n}|detA^{-\frac{1}{2}}(X)|\frac{A^{-1}(X)(X-Y)}{|A^{-\frac{1}{2}}(X)(X-Y)|^n}, \quad (3.7)$$

And for this particular fundamental solution, we have $\forall\ X, Y \in \overline{\Omega}$ :

$$|\nabla_Y K(X,Y) - \nabla_Y K^X(X,Y)| \leq C(\Omega, \Omega', \lambda, A')|X-Y|^{2-n} \quad (3.8)$$

where $A'$ is the Lipschitz constant of the matrix $A$.

Now, let $T_\epsilon$ be the operator defined in (3.2), then we have

$$\begin{aligned} T_\epsilon f(P) &= \int_{\partial\Omega, |P-Q|>\epsilon} \left\langle A(Q)\nabla_Q(K(P,Q) - K^P(P,Q)), N_Q\right\rangle f(Q)dQ \\ &+ \int_{\partial\Omega, |P-Q|>\epsilon} \left\langle (A(Q) - A(P))\nabla_Q K^P(P,Q), N_Q\right\rangle f(Q)dQ \\ &+ \int_{\partial\Omega, |P-Q|>\epsilon} A(P)\nabla_Q K^P(P,Q) N_Q f(Q)dQ \\ &= T_{1,\epsilon}f(P) + T_{2,\epsilon}f(P) + T_{3,\epsilon}f(P) \end{aligned}$$

Clearly $T_i f(P) = \lim_{\epsilon \to 0} T_{i,\epsilon} f(P)$, $i = 1, 2$ exist pointwise a.e. in $\partial\Omega$ and in $L^p(\partial\Omega)$, $1 < p < \infty$, because both operators have integrable kernels (using 3.8 for $T_{1,\epsilon}$ and the fact that the matrix $A$ is Lipschitz for $T_{2,\epsilon}$, both kernels are bounded by $C|P-Q|^{2-n}$, with C depending on $A', \Omega, \Omega', \lambda$, and $n$). Moreover they are compact operators in $L^p(\partial\Omega)$, $1 < p < \infty$.

Now, we only need to analyze $T_{3,\epsilon}$. We put $\tilde{T}_\epsilon = T_{3,\epsilon}$, and using (3.7), we have

$$\tilde{T}_\epsilon f(P) = -\omega_n^{-1} |det A^{-\frac{1}{2}}(P)| \int_{\partial\Omega, |P-Q|>\epsilon} \frac{\langle P - Q, N_Q \rangle}{|A^{-\frac{1}{2}}(P)(P-Q)|^n} f(Q) dQ \quad (3.9)$$

In order to establish the continuity in $L^p(\partial\Omega)$ of this operator, we begin studying the operator

$$\tilde{K}_\epsilon f(x) = \int_{|x-y|>\epsilon} k(x,y) f(y) dy, \quad (3.10)$$

where

$$k(x,y) = \frac{< (x-y, \phi(x) - \phi(y)), (-\nabla\phi(y), 1) >}{|A^{-\frac{1}{2}}(x, \phi(x))(x-y, \phi(x) - \phi(y))|^n}. \quad (3.11)$$

And we have the following result:

**Theorem 3.2** *Assume $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant $M < \infty$. Let $k$ be the kernel in (3.11) and $\tilde{K}_\epsilon f$ the operator defined in (3.10). Then the mapping $\tilde{K}^* f(x) = \sup_{\epsilon>0} |\tilde{K}_\epsilon f(x)|$ is bounded in $L^2(\mathbb{R}^{n-1})$. Furthermore $\lim_{\epsilon\to 0} \tilde{K}_\epsilon f(x) = \tilde{K} f(x)$ exists pointwise a.e. and is a Calderon-Zygmund operator. Moreover when $\phi \in C^1$, $\tilde{K}$ is a compact operator in $L^p(\mathbb{R}^{n-1})$, $1 < p < \infty$.*

**Proof.** By taking into account the next theorem, the proof follows as in the case $A = I$.

**Theorem 3.3** *Let $\phi$ and $\eta$ be two Lipschitz functions in $\mathbb{R}^{n-1}$, and $B(X)$ an $n \times n$ matrix, uniformly elliptic with Lipschistz coefficients, i.e there exist $\mu$ and $\lambda$, such that $\mu|\xi|^2 \leq \langle B(X)\xi, \xi \rangle \leq \lambda|\xi|^2$, uniformly in $X \in \mathbb{R}^n$, and $|B_{i,j}(X) - B_{i,j}(Y)| \leq B'|X-Y|, \forall\ X, Y \in \mathbb{R}^n$, $\forall i, j = 1, ..., n$. Let*

$$K_\epsilon(B) f(x) = \int_{|x-y|>\epsilon} \frac{\eta(x) - \eta(y)}{|B(x, \phi(x))(x-y, \phi(x) - \phi(y))|^n} f(y) dy$$

*Then $K^*(B) f(x) = \sup_{\epsilon>0} |K_\epsilon(B) f(x)|$ is a bounded operator in $L^2(\mathbb{R}^{n-1})$. Moreover, there exists $\lim_{\epsilon\to 0} K_\epsilon(B) f(x) = K(B) f(x)$ a.e. $x \in \mathbb{R}^{n-1}$ and in $L^2$, then $K(B)$ is a Calderón-Zygmund operator in $\mathbb{R}^{n-1}$.*

**Proof.** The proof involves very cumbersome technicalities and will be included as an Appendix.

**Theorem 3.4** *Let $\bar{T}_\epsilon$ be the operator defined by (3.9) in a Lipschitz domain $\Omega$. Then $\bar{T}^* f = \sup_{\epsilon>0} |\bar{T}_\epsilon f|$ is bounded in $L^2(\partial\Omega)$, there exists $\bar{T} f(P) = \lim_{\epsilon\to 0} \bar{T}_\epsilon f(P)$ pointwise a.e. and $\bar{T}$ is a Calderón-Zygmund operator.*
*When $\Omega$ is a $C^1$ domain, $\bar{T}$ is a compact operator in $L^p(\partial\Omega)$, $1 < p < \infty$.*

**Proof.** By means of a partition of unity argument and by passing to local coordinates, the theorem is readily seen to follow from the corresponding statements for the Euclidean operator

$$-\omega_n^{-1} |det A^{-\frac{1}{2}}(x, \phi(x))| K_\epsilon f(x) \tag{3.12}$$

where

$$K_\epsilon f(x) = \int_{\mathcal{U}_\epsilon} k(x, y) f(y) dy, \tag{3.13}$$

with $k(x, y)$ the kernel defined by (3.11), $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ a Lipschitz function as in definition 3.1, and $\mathcal{U}_\epsilon = \{y \in \mathbb{R}^{n-1} : |x-y|^2 + (\phi(x) - \phi(y))^2 > \epsilon^2\}$. By the hipotheses on the matrix $A$, we see $\omega_n^{-1} |det A^{-\frac{1}{2}}(P)| \in L^\infty(\partial\Omega)$, then it's enough to proof the theorem for the operators $K_\epsilon$. We begin showing that

$$\sup_{\epsilon>0} |K_\epsilon f(x) - \bar{K}_\epsilon f(x)| \leq C M f(x), \quad x \in \mathbb{R}^{n-1} \tag{3.14}$$

where $\bar{K}_\epsilon$ is the operator in (3.10), $Mf(x)$ the Hardy-Littlewood maximal function, and C is an absolute constant which depends only on the Lipschitz constant of $\phi$ and on the ellipticity constant $\lambda$. Let

$$\Delta_\epsilon f(x) = K_\epsilon f(x) - \bar{K}_\epsilon f(x) = \int_{R(\epsilon)} k(x, y) f(y) dy \tag{3.15}$$

where $R(\epsilon) = \{y \in \mathbb{R}^{n-1} : |x-y| \leq \epsilon$, and $|x-y|^2 + |\phi(x) - \phi(y)|^2 > \epsilon^2\}$. For $y \in R(\epsilon)$, we have $|k(x,y)| \leq C\epsilon^{1-n}$, with C depending only on $M$ and $\lambda$, so

$$\sup_{\epsilon>0} |\Delta_\epsilon f(x)| \leq C \sup_{\epsilon>0} \epsilon^{1-n} \int_{|x-y|\leq\epsilon} |f(y)| dy \leq C M f(x) \tag{3.16}$$

and (3.14) follows. In turn, (3.14) gives $K^* f(x) \leq \bar{K}^* f(x) + C M f(x)$, and consequently by theorem 3.2, $\|K^* f(x)\|_2 \leq c \|f\|_2$ as we wanted to show. In order to prove the existence of the principal value $Kf(x)$, we shall see that $\lim_{\epsilon\to 0}\Delta_\epsilon f(x) = 0$ pointwise a.e.. It's enough to see it when $f$ is a Lipschitz function. We return to the notation $P = (x, \phi(x))$, $Q = (y, \phi(y))$, $N_Q = (-\nabla\phi(y), 1)$, $dQ = dy$. Now, almost every $x \in \mathbb{R}^{n-1}$ is a Lebesgue point for $N_Q$, i.e. $\lim_{\epsilon\to 0}\epsilon^{1-n}\int_{|x-y|\leq\epsilon}|N_Q - N_P|dQ = 0$. Then replacing for all Lebesgue point of $N_Q$, $N_Q$ by $N_P$, we only need to see

$$\lim_{\epsilon\to 0}\int_{R(\epsilon)}\frac{P-Q}{|A^{-\frac{1}{2}}(P)(P-Q)|^n}dQ = 0. \tag{3.17}$$

for every $x$ where $\phi$ is differentiable. And this statement follows in the same way that in the Laplacian case, see for instance, Coifman-Meyer [8]. Then $Kf(x)$ exists, and

$$\begin{aligned} Kf(x) &= \lim_{\epsilon\to 0}\bar{K}_\epsilon f(x) + \lim_{\epsilon\to 0}\Delta_\epsilon f(x) \\ &= \bar{K}f(x), \text{ pp } x \in \mathbb{R}^{n-1} \end{aligned}$$

And obviously we have also the compactness of $K$ in $L^p(\mathbb{R}^{n-1})$ when $\varphi \in C^1$. □

And then, we have stated all the results that are necessary for the proof of Theorem 3.1.

Next we consider the behavior of the double layer potential $\mathcal{T}f(X)$ given by (3.1) for $X$ near the boundary $\partial\Omega$. Since the notion of nontangential convergence is the appropriate here, we begin by defining inner and outer cones to $\Omega$.

**Definition 3.2** *Given $P \in \partial\Omega$, $\Gamma(P)$ will denote the doublely truncated cone, with two convex components, non empty, with vertex at $P$, one component in $\Omega$ and the other one in $\mathbb{R}^n \setminus \bar{\Omega}$.*

*Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function as in Definition 3.1, with Lipschitz constant $M$. If we note $X = (x, t)$ the points in $\Omega$ with $x \in \mathbb{R}^{n-1}$, and $t \in \mathbb{R}$, and $P = (x_0, \phi(x_0))$ a fixed point in $\partial\Omega$, we can assume that $\Omega$ is globally defined by $t > \phi(x)$. Let $a > M$, then for each $P \in \partial\Omega$ we have a nontangential cone, totally included in $\Omega$*

$$\Gamma_a(P) = \{(x,t) : t > \phi(x), t - \phi(x_0) \geq a|x - x_0|\} \cap B(P, \tau) \tag{3.18}$$

*with $\tau$ a constant depending only on $a$ and $\Omega$. Similarly we can define the outer cones as*

$$\Gamma_a^e(P) = \{(x,t) : t < \phi(x), t - \phi(x_0) \leq -a|x - x_0|\} \cap B(P, \tau) \tag{3.19}$$

**Theorem 3.5** *Let $\mathcal{T}f(X)$ be the double layer potential corresponding to a Lipschitz domain $\Omega$ given by (3.1), and let consider*

$$N_a u(P) = \sup_{X \in \Gamma_a(P)} |u(X)| \tag{3.20}$$

*($N_a^e u(P) = \sup_{X \in \Gamma_a^e(P)} |u(X)|$). Then if $f \in L^p(\partial\Omega)$, $1 < p < \infty$, given $a$ as in definition 3.2. we have*

$$\|N_a(\mathcal{T}f)\|_p, \|N_a^e(\mathcal{T}f)\|_p \leq c\|f\|_p, \quad 1 < p < \infty \tag{3.21}$$

*where $c$ depends only on $p, a$, and $\lambda$. Furthermore, for almost every $P \in \partial\Omega$*

$$\lim_{X \in \Gamma_a(P), X \to P} \mathcal{T}f(X) = (T - \frac{1}{2})f(P) \tag{3.22}$$

$$\lim_{X \in \Gamma_a^e(P), X \to P} \mathcal{T}f(X) = (T + \frac{1}{2})f(P) \tag{3.23}$$

*where $Tf(P)$ is the trace double layer potential given by (3.2).*

**Proof.** As in Theorem 3.1 we only need to study the operator

$$\mathcal{T}_3 f(X) = \int_{\partial\Omega} \left\langle A(X)\nabla_Q K^X(X, Q), N_Q \right\rangle f(Q)dQ,$$

where $K^X(X, Y)$ is the function given by (3.6).

Let $\phi$ be a Lipschitz function as in Definition 3.1, we denote $X = (x, t)$, with $t > \phi(x)$ the points in $\Omega$, and $Q = (y, \phi(y))$ or $P = (x_0, \phi(x_0))$ the points in $\partial\Omega$ (depending on the context we shall use the different notations). Now, let $a$ and $M$ be constants such that $a > M > \|\nabla\phi\|_\infty$, $\Gamma_a(P)$, and $\Gamma_a^e(P)$ the inner and outer cones defined by (3.18), and (3.19) respectively. By the definition of $K^X(X, Y)$, the kernel of the operator $\mathcal{T}_3$ is

$$\frac{|det A^{-\frac{1}{2}}(x,t)|}{-\omega_n}\left(\frac{t - \phi(y) - \langle (x-y), \nabla\phi(y)\rangle}{|A^{-\frac{1}{2}}(x,t)(x-y, t-\phi(y))|^n}\right)$$

Now, taking $\epsilon = t - \phi(x_0)$, for $(x,t) \in \Gamma_a(P)$

$$\frac{t - \phi(y) - (x-y)\nabla\phi(y)}{|A^{-\frac{1}{2}}(x,t)(x-y, t-\phi(y))|^n} = k_\epsilon(\phi, x_0, y) + R_\epsilon(x_0, x, y) \tag{3.24}$$

where

$$k_\epsilon(\phi, x, y) = \begin{cases} k(x,y), & \text{if } |x-y| > \epsilon \\ 0, & \text{if } |x-y| \leq \epsilon \end{cases} \tag{3.25}$$

and $k(x,y)$ is given by (3.11). Moreover there exists $C = C(M, a, \lambda, A')$, such that

$$|R_\epsilon(x_0, x, y)| \leq C\epsilon^{1-n}, \text{ for } |x_0 - y| \leq \epsilon \tag{3.26}$$

and

$$|R_\epsilon(x_0, x, y)| \leq C\epsilon|x_0 - y|^{-n}, \text{ for } |x_0 - y| \geq \epsilon \tag{3.27}$$

The proof of these estimates is merely technical, then we have omitted it, but it follows esentially by taking into account that the matrix $A$ is uniformly elliptic and it has Lipschitz coefficients, all the details are in [10].

By other hand, $\epsilon \int_{|x_0-y|>\epsilon} |x_0 - y|^{-n}|f(y)|dy \leq C_n Mf(x_0)$, where $Mf(x_0)$ is the maximal function of Hardy-Littlewood. Then

$$N_a(\mathcal{T}_3 f)(P) \leq C_{\lambda,n}(T^* f(P) + Mf(P)) \tag{3.28}$$

where $T^* f(P) = \sup_{\epsilon>0} |\int k_\epsilon(\phi, x, y) f(y) dy|$. Now by Theorem 3.2,

$$||N_a(\mathcal{T}_3 f)||_p \leq C_{n,\lambda}||f||_p, \quad \forall 1 < p < \infty$$

and (3.21) is proved.

To see the nontangential boundary values of $\mathcal{T}f(P)$ we consider $f \in Lip(\partial\Omega)$, then

$$\begin{aligned}
\mathcal{T}f(X) &= \int_{\partial\Omega} \langle A(Q)\nabla_Q K(X,Q), N_Q\rangle [f(Q) - f(P)]dQ \\
&+ f(P)\int_{\partial\Omega} \langle A(Q)\nabla_Q K(X,Q), N_Q\rangle dQ \\
&= I_1 + I_2.
\end{aligned}$$

Since $|f(Q) - f(P)| \leq C|Q - P|$, the integrand of $I_1$ has a summable singularity and the limit exists. and it is $Tf(P) + 1/2f(P)$, and the integral in $I_2$ is $\equiv -1$, $\forall\ X \in \Omega$ (both results follow in the same way that for the Laplacian operator. essentially because we can apply Green's theorem, the technical details are in [10]), then we have

$$\lim_{X\in\Gamma(P), X\to P} \mathcal{T}f(X) = (T - \frac{1}{2})f(P)$$

and taking the limit by outer cones we have :

$$\lim_{X \in \Gamma^e(P), X \to P} Tf(X) = (T + \frac{1}{2})f(P)$$

By the continuity in $L^p$ of the nontangential maximal function, we have the continuity for all $f \in L^p$. □

# 4 The single layer potential. Its gradient.

In this section we are interested in the nontangential limits of the gradient of the single layer potential defined by (3.4), when $X$ approaches the boundary.

We will need the following result:

**Theorem 4.1** *Let $T^t_\epsilon f(P)$ be the potential corresponding to a Lipschitz domain defined as*

$$T^t_\epsilon f(P) = \int_{\partial\Omega, |P-Q|>\epsilon} \langle A(P)\nabla_P K(Q,P), N_P\rangle f(Q)dQ, \ P \in \partial\Omega. \quad (4.1)$$

*Then the mapping $(T^t)^* f(P) = \sup_{\epsilon>0} |T^t_\epsilon f(P)|$ is bounded in $L^2(\partial\Omega)$, $T^t f(P) = \lim_{\epsilon\to 0} T^t_\epsilon f(P)$, exists pointwise a.e. and $T^t$ is a Calderón-Zygmund operator. Moreover, when $\partial\Omega \in C^1$, $T^t f(P)$ is compact in $L^p(\partial\Omega)$.*

**Proof.** We only need to prove that $\lim_{\epsilon\to 0} T^t_\epsilon f(P)$ exists, the other statements follow from Theorem 3.1 (because in this case, the operator $T^t$ is the adjoint operator of $T$). As in this theorem we compare the fundamental solution for $L$ with the fundamental solution for $L^Y u(X) = -div(A(Y)\nabla u(X))$, given by (3.6). Now, the relevant operator has distributional kernel

$$\left\langle A(Q)\nabla_P K^Q(Q,P), N_P \right\rangle,$$

and the proof is almost the same that in Theorem 3.4. □

From now on, in order to do the notation easier, we shall consider **symmetric fundamental solutions**.

**Theorem 4.2** *Let $\Omega$ be a Lipschitz domain, for $f \in L^p(\partial\Omega)$, $1 < p < \infty$, and $X \notin \partial\Omega$, let $Sf(X)$ be the single layer potential given by (3.4). Then there exist numbers $a$ depending only on $\Omega$, so that $N_a(|\nabla Sf|)(P)$ and $N^e_a(|\nabla Sf|)(P)$ belong to $L^p(\partial\Omega)$, and there is a constant $c$ which depends only on $p, a$, and $\lambda$ so that*

$$\|N_a(|\nabla Sf|)\|_p, \ \|N^e_a(|\nabla Sf|)\|_p \le c\|f\|_p, \ 1 < p < \infty \quad (4.2)$$

*Furthermore, $\lim_{X\in\Gamma_a(P), X\to P} \langle A(P)\nabla Sf(X), N_P\rangle = (T^t + \frac{1}{2})f(P)$ and*

$\lim_{X\in\Gamma_a^e(P), X\to P} \langle A(P)\nabla \mathcal{S}f(X), N_P\rangle = (T^t - \frac{1}{2})f(P)$ *exist pointwise for almost every* $P \in \partial\Omega$.

**Proof.** Observe that

$$\nabla \mathcal{S}f(X) = \int_{\partial\Omega} \nabla_X K(X,Q)f(Q)dQ,$$

then the proof of estimates (4.2) follow along the lines of Theorem (3.5) and is therefore omitted. For the nontangential convergence it suffices to prove the existence of the pointwise limit for almost every $P \in \partial\Omega$ when $f \in Lip(\partial\Omega)$. The proof is again taking fundamental solutions for elliptic operators with constant coefficients. Let $K^{X_0}(X,Y)$ be the fundamental solution for $L^{X_0} = -div(A(X_0)\nabla)$ given by (3.6), then the single layer potential in this case is given by

$$\mathcal{S}^{X_0}f(X) = \int_{\partial\Omega} K^{X_0}(X,Q)f(Q)dQ.$$

And we know that

$$\lim_{X\in\Gamma_a(P), X\to P} \left\langle A(X_0)\nabla \mathcal{S}^{X_0}f(X), N_P\right\rangle = \frac{1}{\omega_n}|detA^{-\frac{1}{2}}(X_0)|vp\int_{\partial\Omega} \frac{\langle (P-Q), N_P\rangle}{|A^{-\frac{1}{2}}(X_0)(P-Q)|^n} f(Q)dQ + \frac{1}{2}f(P) \tag{4.3}$$

Now, returning to the gradient of the single layer potential for $L$, we write

$$\begin{aligned} \langle A(P)\nabla \mathcal{S}f(X), N_P\rangle &= \int_{\partial\Omega} \langle A(P)\nabla_X K(X,Q), N_P\rangle f(Q)dQ \\ &= \int_{\partial\Omega} \left\langle A(P)(\nabla_X K(X,Q) - \nabla_X K^P(X,Q)), N_P\right\rangle f(Q)dQ \\ &+ \int_{\partial\Omega} \left\langle A(P)\nabla_X K^P(X,Q), N_P\right\rangle f(Q)dQ \\ &= I_1 + I_2 \end{aligned} \tag{4.4}$$

Clearly, as the limit in (4.3) exists pointwise for almost all $P \in \partial\Omega$, and as the matrix $A$ is defined in all $\overline{\Omega}$, we can take $X_0 = P$, then

$$\begin{aligned} \lim_{\{X\in\Gamma(P), X\to P\}} I_2 &= \omega_n{}^{-1}|detA^{-\frac{1}{2}}(P)|vp\int_{\partial\Omega} \frac{\langle (P-Q), N_P\rangle}{|A^{-\frac{1}{2}}(P)(P-Q)|^n} f(Q)dQ \\ &+ \frac{1}{2}f(P). \end{aligned} \tag{4.5}$$

Now, $\forall\ Q \in \partial\Omega$ and $\forall\ X \in \Gamma_a(P)$, by the properties of the matrix $A$ and of the fundamental solutions involved, we have that

$$|\nabla_X K(X,Q) - \nabla_X K^P(X,Q)| \leq C|P-Q|^{2-n} \tag{4.6}$$

where C is a constant depending on $\lambda, n, a$ and on the Lipschitz constant for $A$. Then by Lebesgue dominated convergence,

$$\lim_{X\to P} I_1 = \int_{\partial\Omega} \left\langle A(P)(\nabla_X K(P,Q) - \nabla_X K^P(P,Q)), N_P \right\rangle f(Q)dQ \tag{4.7}$$

Now, by (4.7), and (4.5)

$$\lim_{X\in\Gamma_a(P), X\to P} \langle A(P)\nabla \mathcal{S}f(X), N_P\rangle = (T^t + \frac{1}{2})f(P) \tag{4.8}$$

and in the same way,

$$\lim_{\{X\in\Gamma_a^c(P), X\to P\}} \langle A(P)\nabla \mathcal{S}f(X), N_P\rangle = (T^t - \frac{1}{2})f(P) \tag{4.9}$$

□

# 5 Invertibility theorems. Dirichlet and Neumann problems.

## 5.1 Invertibility theorems on $L^p(\partial\Omega)$.

Let us first describe under which conditions we can construct the inverse operators.

We consider the operator $L = -div_Y(A(Y)\nabla)$ in a bounded and $C^1$ domain $\Omega' \subset \mathbb{R}^n, n \geq 3$, where $A(Y)$ is a real, symmetric, $n \times n$ matrix with Lipschitz coefficients in $\Omega'$ and uniformly elliptic, and let $\Omega$ be a $C^1$ domain such that $\overline{\Omega} \subset \Omega'$. Henceforth, we will take as fundamental solution, **the Green function for $L$ in $\Omega'$**, i.e., the symmetric and positive fundamental solution vanishing on $\partial\Omega'$, and we will note it by $G'(X,Y)$ (also we can take a fundamental solution with the same behavior at the infinity). In the following sections, we will note by $\mathcal{G}$ and $G$, the double layer potential and its trace, related to the fundamental solution $G'$.

**Remark.** Some proofs in this section make use of some ideas developed in [12].

**Theorem 5.1** *Assume $\Omega$ is a bounded $C^1$ domain and $\Omega' \setminus \overline{\Omega}$ is connected. Let $Gf(P)$ denote the trace double layer potential defined by (3.3) with $K(X,Y) = G'(X,Y)$. Then $G - \frac{1}{2}I$ is invertible on $L^p(\partial\Omega)$ for each $1 < p < \infty$.*

**Proof.** We show in fact that the adjoint of $G - \frac{1}{2}I$, i.e. $G^t - \frac{1}{2}I$, is invertible on $L^p(\partial\Omega)$. Since, by Theorem 4.1, $G^t$ is compact in $L^p(\partial\Omega)$, it is enough to prove that $G^t - \frac{1}{2}I$ is injective (by Fredholm's theory). First we observe that if $f \in L^p(\partial\Omega)$, and $(G^t - \frac{1}{2}I)f = 0$, then $f \in L^q(\partial\Omega)\ \forall\ q : 1 < q < \infty$. (the proof is no different than that for $L = \Delta$, in [4]). Now consider the single layer potential of the function $f$ over $\partial\Omega$ i.e.,

$$\mathcal{S}f(X) = \int_{\partial\Omega} G'(X,Y)f(Y)dY \tag{5.1}$$

By the ellipticity condition and applying the Green's theorem , we have

$$\begin{aligned} \lambda\int_{\Omega'\setminus\Omega} |\nabla\mathcal{S}f(X)|^2 dX &\leq \int_{\Omega'\setminus\Omega} A(X)\nabla\mathcal{S}f(X))\nabla\mathcal{S}f(X)dX \\ &\leq -\int_{\partial\Omega} \partial^e\eta_A\mathcal{S}f(Q)\mathcal{S}f(Q)dQ = 0 \end{aligned} \tag{5.2}$$

where $(\partial^e\eta_A) = \lim_{X\in\Gamma^e(Q),X\to Q}\langle A(Q)\nabla\mathcal{S}f(X), N_Q\rangle$, and we have used the fact that $\mathcal{S}f(X)$ is a solution in $\Omega'\setminus\Omega$, and $\mathcal{S}f(P) \equiv 0$ in $\partial\Omega'$. By Theorem 4.2, the last integral in (5.2) is absolutely convergent and $(\partial^e\eta_A)\mathcal{S}f(Q) = (G^t - \frac{1}{2}I)f(Q) = 0$ a.e. in $\partial\Omega$. Therefore $\mathcal{S}f(X)$ is constant on $\Omega'\setminus\Omega$, and since $\mathcal{S}f = 0$ on $\partial\Omega'$, by continuity $\mathcal{S}f(X) \equiv 0$ in $\Omega'\setminus\Omega$ and $\mathcal{S}f \equiv 0$ on $\partial\Omega$. Now, in the same way as for $\Omega'\setminus\Omega$, we can see $\mathcal{S}f(X)$ is constant on $\Omega$. Thus $\mathcal{S}f(X) \equiv 0$ on all $\Omega'$ so that $\partial\eta_A\mathcal{S}f(Q) = (G^t + \frac{1}{2}I)f(Q) = 0$ for almost every $Q \in \partial\Omega$, and we conclude for these $Q$, $f(Q) = (\frac{1}{2}I + G^t)f(Q) - (G^t - \frac{1}{2}I)f(Q) = 0$. □

**Theorem 5.2** *Assume $\Omega$ is a bounded, connected, $C^1$ domain, and let $G^t$ be the operator defined in Theorem 5.1. Then for each $1 < p < \infty$, $G^t + \frac{1}{2}I$ is invertible in $L^p_0(\partial\Omega)$, where*

$$L^p_0(\partial\Omega) = \{f \in L^p(\partial\Omega)\ :\ \int_{\partial\Omega} f(Q)dQ = 0\}.$$

**Proof.** Since the single layer potential is a solution for $Lu = 0$ in $\Omega$, and we can apply the Green's theorem, we have $\int_{\partial\Omega} \partial\eta_{A(P)}\mathcal{S}f(P)dP = \int_{\partial\Omega}(\frac{1}{2}I + G^t)f(P)dP = 0$.

For the invertibility on $L^p_0(\partial\Omega)$, again by the compactness of the operator $G^t$, it is enough to prove that $G^t + \frac{1}{2}I$ is injective. As in the previous theorem, if we choose $f \in L^p(\partial\Omega)$ such that $f = -2G^tf$, and $\int_{\partial\Omega} f(Q)dQ = 0$, then $f \in L^q(\partial\Omega)$, $\forall\ 1 < q < \infty$.

Now, integrating by parts we get,

$$\int_\Omega |\nabla\mathcal{S}f(X)|^2 dX \leq \lambda^{-1}\int_{\partial\Omega}(G^t + \frac{1}{2}I)f(Q)\mathcal{S}f(Q)dQ = 0.$$

Hence $\mathcal{S}f(X)$ is constant in $\Omega$. By other hand, by the hypothesis over $f$ we have $\int_{\partial\Omega}(-\frac{1}{2}I + G^t)f(Q)dQ = -\int_{\partial\Omega} f(Q)dQ = 0$. then

$$\int_{\Omega'\setminus\Omega} |\nabla \mathcal{S}f(X)|^2 dX \leq \lambda^{-1}C\int_{\partial\Omega}(G^t - \frac{1}{2}I)f(Q)dQ = 0$$

By definition $\mathcal{S}f(P) = 0$ on $\partial\Omega'$. then we conclude that $\mathcal{S}f = 0$ on $\Omega'$. and then $f(P) = 0$ on $\partial\Omega$ (as we have seen in Theorem ). so $G^t + \frac{1}{2}I$ is injective on $L_0^p(\partial\Omega)$. □

## 5.2 Dirichlet and Neumann problems

Once we have proved the previous theorems. the proof of the existence and uniqueness of the solutions to the Dirichlet and Neumann problems with boundary values in $L^p(\partial\Omega)$ follow in the same way that the proof by Fabes. Jodeit and Riviere in [4]. when $L = \Delta$. Then we only mention the statements and how the solutions may be represented by potentials of $L^p(\partial\Omega)$ functions.

**Theorem 5.3** *Suppose $\Omega$ is a $C^1$ domain and $R^n \setminus \overline{\Omega}$ is connected. Given $f \in L^p(\partial\Omega))$. $1 < p < \infty$, there exists a unique function $u(X)$ defined in $\Omega$ such that $Lu = 0$. and*

- *$N_a(u)$ (defined in (3.20)) belongs to $L^p(\partial\Omega)$. and $||N_a(u)||_p \leq c||f||_p$. with c independent of $f$, moreover, a.e on $\partial\Omega$ $\lim_{X\to P, X\in\Gamma(P)} u(X) = f(P)$*
- *$u(X)$ is the double layer potential of $(G - \frac{1}{2}I)^{-1}f(P)$ over $\partial\Omega$, defined by mean of the Green function defined before, i.e*

$$u(X) = \int_{\partial\Omega} \langle A(Q)\nabla_Q G'(X,Q), N_Q\rangle (G - \frac{1}{2}I)^{-1} f(Q)dQ$$

**Theorem 5.4** *Suppose $\Omega$ is a bounded, connected. $C^1$ domain. and $R^n \setminus \overline{\Omega}$ is connected. Given $g \in L^p(\partial\Omega))$, $1 < p < \infty$, with $\int_{\partial\Omega} g(Q)dQ = 0$. there exists a function $u(X)$ defined for $X \in \Omega$ such that $Lu = 0$. and*

- *The nontangential maximal function (defined by (3.20) of $\nabla u$ belongs to $L^p(\partial\Omega)$. and we have $||N_a(|\nabla u|)||_p \leq c||g||_p$ with c independent of $g$. Moreover*

$$\lim_{X\to P, X\in\Gamma(P)} \langle A(P)\nabla u(X). N_P\rangle = g(P)$$

- *The function $u(X)$ is uniquely determined up to a constant and can be taken as the single layer potential of $(G^t + \frac{1}{2}I)^{-1}g(Q)$ over $\partial\Omega$, i.e.*

$$u(X) = \int_{\partial\Omega} G'(X, Q)(G^t + \frac{1}{2}I)^{-1}g(Q)dQ.$$

## 5.3 Regularity theorems

**Definition 5.1** *$f \in L_1^p(\partial\Omega)$, $1 < p < \infty$ if $f \in L^p(\partial\Omega)$, and $f(x, \phi(x))$ has distributional gradient in $L^p(R^{n-1})$ (where $\phi(x)$ is given by Definition 3.1). It is easy to check that if $F$ is any extension to $R^n$ of $f$, then $\nabla_x F(x, \phi(x))$ is well defined and belongs to $L^p(\partial\Omega)$. We call this $\nabla_t f$. Then*

$$||f||_{L_1^p(\partial\Omega)} = ||f||_p + ||\nabla_t f||_p$$

**Theorem 5.5** . *Let $Sf(P)$ be the trace single layer potential over a Lipschitz domain $\Omega$. then $\forall p$ : $1 < p < \infty$ $S : L^p(\partial\Omega) \to L_1^p(\partial\Omega)$ is a bounded operator. Furthermore, if $\Omega$ is a $C^1$ domain, then $S$ has bounded inverse on $L^2(\partial\Omega)$.*

**Proof**. By Theorem 4.2 (really from the proof of this Theorem) we also get the existence and the continuity of the nontangential limits of the gradient of $S$, by means of Theorem 3.3.

As in Theorems 5.1 and 5.2 we see that $Sf(P) = 0$ a.e. on $\partial\Omega$ implies $f = 0$ a.e. on $\partial\Omega$, so that $S$ is one to one.

Since $\frac{1}{2} + G^t : L^p(\partial\Omega) \to L_0^p(\partial\Omega)$ and is invertible on the latter, there exists a unique function $f_0$, in the kernel of this operator such that $\int_{\partial\Omega} f_0(Q)dQ = 1$, now $Sf_0$ is constant on $\Omega$, and the constant is not equal to zero since $S$ is one to one.

In order to prove that $S$ is onto we take $f \in L_1^2(\partial\Omega)$ and we consider $u$ such that $Lu = 0$ in $\Omega$ and $u = f$ on $\partial\Omega$, then $\partial\eta_A u(P)$ belongs to $L^2(\partial\Omega)$ (by Rellich identity, see for instance J. Nečas [9]).

Now, by Theorem 4.2

$$\partial\eta_A S(\frac{1}{2}I + G^t)^{-1}\partial\eta_A u(P) = \partial\eta_A u(P).$$

then $S(\frac{1}{2}I + G^t)^{-1}\partial\eta_A u(X) = u(X) + c$, by Theorem 5.4, but we have shown that the constants are in the rank of $S$, then we have $f(P) = S((\frac{1}{2}I + G^t)^{-1}\partial\eta_A u + f_c)(P)$, for almost every $P \in \partial\Omega$. Then $S : L^2(\partial\Omega) \to L_1^2(\partial\Omega)$ has bounded inverse. □

**Theorem 5.6** . *When $\Omega$ is a $C^1$ domain and $1 < p < \infty$. $\frac{1}{2}I - G : L_1^p(\partial\Omega) \to L_1^p(\partial\Omega)$ is an invertible operator.*

**Proof.** Given $g \in L_1^2(\partial\Omega)$, we take $v(X) = \mathcal{S}(\frac{1}{2}I - G^t)^{-1}S^{-1}g(X)$. By Theorem 4.2. for almost every $P \in \partial\Omega$

$$\lim_{X \to P, X \in \Gamma^e(P)} \langle A(P)\nabla v(X), N_P \rangle = S^{-1}g(P).$$

obviously $v$ is a solution for $Lv = 0$ in $\Omega$, and $v = 0$ on $\partial\Omega'$. then we can extend this function as 0 outside $\Omega'$, and taking $X \in \Omega' \setminus \Omega$ we have

$$\begin{aligned}
v(X) &= \int_{\Omega'} A(Y)\nabla_Y G'(X,Y)\nabla_Y v(Y)dY \\
&= \int_{\Omega' \setminus \Omega} A(Y)\nabla_Y G'(X,Y)\nabla_Y v(Y)dY + \int_{\Omega} A(Y)\nabla_Y G'(X,Y)\nabla_Y v(Y)dY \\
&= \int_{\partial(\Omega' \setminus \Omega)} \partial\eta_A v(Q)G'(X,Q)dQ + \int_{\Omega' \setminus \Omega} G'(X,Y)Lv(Y)dY \\
&+ \int_{\Omega} L_Y G'(X,Y)v(Y)dY + \int_{\partial\Omega} \partial\eta_A G'(X,Q)v(Q)dQ \\
&= -\mathcal{S}(S^{-1}g)(X) + \mathcal{G}f(X)
\end{aligned}$$

where $f(P)$ are the boundary values of $v$. Taking nontangential limit by outer cones to $\Omega$. we obtain for almost every $P \in \partial\Omega$

$$f(P) = -g(P) + (\frac{1}{2}I + G)v(P).$$

then $(G - \frac{1}{2}I)f(P) = g(P)$, and the operator is onto $L_1^2(\partial\Omega)$. Moreover, every $f \in L_1^2(\partial\Omega)$ can be written as $S(G^t - \frac{1}{2}I)^{-1}S^{-1}g$. for any $g \in L_1^2(\partial\Omega)$, following the previous proof we see $(G - \frac{1}{2}I)f(P) = g(P)$, then

$$S(G^t - \frac{1}{2}I)S^{-1} = G - \frac{1}{2}I. \qquad (5.3)$$

and we have proved not only the continuity but also the invertibility from $L_1^2(\partial\Omega)$ into $L_1^2(\partial\Omega)$.

In order to prove this result for all $1 < p < \infty$, we follow the same idea as in Theorem 3.1, but taking into account that given $P \in \partial\Omega, Gf(X) = G(f(X) - f(P)) + cf(P)$ and that using a local representation in formula (2.5). we get also the following estimate for the function $u_X(Y)$:

$$|\partial_{X_j}\partial_{Y_i}u_X(Y)| \leq C_{n,\lambda}R^{1-n}, \text{ when } R \leq |X - Y| \leq 2R. \qquad (5.4)$$

Now, by means of the ideas developped in section 3, the proof follows in the same way as for the Laplacian operator [4]. □

**Corollary 5.1** *Let $Sf(P)$ be the trace single layer potential over a $C^1$ domain $\Omega$, then $\forall p$ $1 < p < \infty$ $S : L^p(\partial\Omega) \to L^p_1(\partial\Omega)$ is an invertible operator*

**Proof.** Following the proof of Theorem 5.5, we only have to see that given $f \in L^p_1(\partial\Omega)$, if we we consider $u$ such that $Lu = 0$ in $\Omega$ and $u = f$ on $\partial\Omega$, then $\partial\eta_A u(P)$ belongs to $L^p(\partial\Omega)$.Now, by Theroem 5.6, this solution is given by $u(X) = \mathcal{G}((G - \frac{1}{2})^{-1}f)(X)$. □

**Corollary 5.2** *Given $f \in L^p_1(\partial\Omega)$, $1 < p < \infty$, for almost every $P \in \partial\Omega$ there exists $(\partial\eta_{A(P)}\mathcal{G})f(P)$, is a continuous operator from $L^p_1(\partial\Omega)$ to $L^p(\partial\Omega)$, and we have*

$$\lim_{X \to P, X \in \Gamma(P)} \langle A(P)\nabla\mathcal{G}f(X), N_P\rangle = (G^t + \frac{1}{2}I)(G^t - \frac{1}{2}I)S^{-1}f(P)$$

**Proof.** Define $u(X) = \mathcal{G}f(X)$, then by 5.3$\lim_{X \to P, X \in \Gamma(P)} u(X) = (G - \frac{1}{2}I)f(P) = S(G^t - \frac{1}{2}I)S^{-1}f(P)$, by uniqueness of the Dirichlet problem, we have

$$\mathcal{G}f(X) = \mathcal{S}(G^t - \frac{1}{2}I)S^{-1}f(X), \text{ so}$$

$$\begin{aligned}\lim_{X \to P, X \in \Gamma(P)} \langle A(P)\nabla\mathcal{G}f(X), N_P\rangle &= \lim_{X \to P, X \in \Gamma(P)} \left\langle A(P)\nabla\mathcal{S}(G^t - \frac{1}{2}I)S^{-1}f(X), N_P\right\rangle \\ &= (G^t + \frac{1}{2}I)(G^t - \frac{1}{2}I)S^{-1}f(P).\end{aligned}$$

And obviously from this we conclude existence and continuity. □

## 6 The Calderón Projector

As we have seen in the Preliminaries, the construction and the properties of the Calderón Projector follow from the analysis of the potentials, then now we are in conditions to describe this Projector when the elliptic operator is $L = -div(A(Y)\nabla_Y)$, defined in a bounded, connected and $C^1$ domain $\bar{\Omega}$, and the $n \times n$ ($n \geq 3$) matrix $A$ is uniformly elliptic with Lipschitz coefficients. We take as fundamental solution $G'(X, Y)$, the Green function on $\Omega'$, with $\Omega' \subset \bar{\Omega}$. Recall that the Calderón Projector $\mathbf{P} = (\mathbf{P}_{i,j})_{i,j=1,2}$, is given by

$$
\begin{aligned}
\mathbf{P}_{1,1}f(P) &= -\lim_{X\to P, X\in\Gamma(P)} \mathcal{G}f(X)\\
\mathbf{P}_{1,2}f(P) &= \lim_{X\to P, X\in\Gamma(P)} \mathcal{S}f(X)\\
\mathbf{P}_{2,1}f(P) &= -\lim_{X\to P, X\in\Gamma(P)} \partial\eta_{A(P)}\mathcal{G}f(X)\\
\mathbf{P}_{2,2}f(P) &= \lim_{X\to P, X\in\Gamma(P)} \partial\eta_{A(P)}\mathcal{S}f(X)
\end{aligned}
$$

where $f$ is defined on $\partial\Omega$, with $\Omega$ a $C^1$ domain such that $\Omega \subset\subset \Omega'$. And $\mathcal{S}$ and $\mathcal{G}$. are the single and double layer potentials respectively, defined by mean of the Green function $G'(X,Y)$(section 5).

**Theorem 6.1** *Let $\Omega$ be a connected and $C^1$ domain. The operator* $\mathbf{P}$ *defined before has the following properties:*

1. $\mathbf{P} : L_1^p(\partial\Omega) \times L^p(\partial\Omega) \to L_1^p(\partial\Omega) \times L^p(\partial\Omega)$, with $1 < p < \infty$

2. $\mathbf{P} = \begin{pmatrix} \frac{1}{2}I - G & S \\ (G^t + \frac{1}{2}I)(\frac{1}{2}I - G^t)S^{-1} & G^t + \frac{1}{2}I \end{pmatrix}$

3. $Im\mathbf{P} = \mathcal{U} = \{(u|_{\partial\Omega}, \partial\eta_A u|_{\partial\Omega}) : Lu = 0 \text{ in } \Omega, u|_{\partial\Omega} \in L_1^p(\partial\Omega), \partial\eta_A u|_{\partial\Omega} \in L_0^p(\partial\Omega)\}$

4. $\mathbf{P}^2 = \mathbf{P}$

**Proof.** We summarize here all the statements proved in the previous sections, e.g.,
$\mathbf{P}_{1,1}f(P) = (\frac{1}{2}I - G)f(P)$ and is a Calderón-Zygmund operator by Theorem 3.1, and is a bounded and invertible operator in $L_1^p(\partial\Omega)$ by Theorem 5.6.
$\mathbf{P}_{1,2}f(P) = Sf(P) : L^p(\partial\Omega) \to L_1^p(\partial\Omega)$, is a bounded and invertible operator by Theorem 5.5.
$\mathbf{P}_{2,1}f(P) = (G^t + \frac{1}{2}I)(\frac{1}{2}I - G^t)S^{-1}f(P)$ and is a bounded operator from $L_1^p(\partial\Omega)$ to $L^p(\partial\Omega)$ by Corollary 5.2.
$\mathbf{P}_{2,2}f(P) = (G^t + \frac{1}{2}I)f(P)$ and is a Calderón-Zygmund operator by Theorem 4.2. This operator is invertible on $L_0^p(\partial\Omega)$ by Theorem 5.2.

Obviously, all these equalities are a.e. $P \in \partial\Omega$. And we have described all the properties about the operators involved in $\mathbf{P}$, particularly we have proved the statements 1 and 2 of the Theorem.

We take now $(f,g) \in L^p_1(\partial\Omega) \times L^p(\partial\Omega)$, $1 < p < \infty$, then

$$\mathbf{P}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} (\frac{1}{2}I - G)f(P) + Sg(P) \\ (G^t + \frac{1}{2}I)(\frac{1}{2}I - G^t)S^{-1}f(P) + (G^t + \frac{1}{2}I)g(P) \end{pmatrix}$$

If we put $v(X) = -\mathcal{G}f(X) + Sg(X)$, obviously $Lv = 0$, and $v$ has boundary values given by $\mathbf{P}\begin{pmatrix} f \\ g \end{pmatrix}$, then $Im\mathbf{P} \subset \mathcal{U} = \{(u|_{\partial\Omega}, \partial\eta_A u|_{\partial\Omega}) : Lu = 0 \ in \ \Omega,\ u|_{\partial\Omega} \in L^p_1(\partial\Omega),\ \partial\eta_A u|_{\partial\Omega} \in L^p_0(\partial\Omega)\}$.

Let now see that given $(f,g) \in \mathcal{U}$, we get $\mathbf{P}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$, and it implies that $\mathcal{U} \subset Im\mathbf{P}$, and that $\mathbf{P}^2 = \mathbf{P}$. By Theorems 5.3 and 5.4, for almost every $P \in \partial\Omega$ we have

$$f(P) = S(G^t + \frac{1}{2}I)^{-1}g(P) \tag{6.1}$$

then

$$\begin{aligned} (\frac{1}{2}I - G)f(P) + Sg(P) &= -S(G^t - \frac{1}{2}I)S^{-1}f(P) + S(G^t + \frac{1}{2}I)S^{-1}f(P) \\ &= S[(\frac{1}{2}I - G^t) + (G^t + \frac{1}{2}I)]S^{-1}f(P) \qquad (6.2) \\ &= f(P) \end{aligned}$$

by (5.3) and (6.1), and

$$(G^t + \frac{1}{2}I)(\frac{1}{2}I - G^t)S^{-1}f(P) + (G^t + \frac{1}{2}I)g(P) = g(P)$$

by (5.3), (6.1) and (6.2). □

# 7 Appendix 1.

In this section we state the proof of **Theorem 3.3.**

**Definition 7.1**
*Let be $Tf(x) = \int k(x,y)f(y)dy$, we say that the kernel $k(x,y)$ satisfies "standard estimates" if it is a continuous function in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\}$, such that there exist constants $C$, and $\delta$, with*

$$|k(x,y)| \leq C|x-y|^{-n}$$

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq C|x - x'|^{\delta}|x - y|^{-(n+\delta)},$$

*if* $|x - x'| \leq \frac{1}{2}|x - y|$. *If* $\delta = 1$, *this second condition is equivalent to*

$$|\nabla_x k(x,y)| + |\nabla_y k(x,y)| \leq C|x - y|^{-1-n}$$

**Definition 7.2**
*Let be* $T : D(\mathbb{R}^n) \to D'(\mathbb{R}^n)$ *a linear and continuous operator, we say that* $T$ *is a Calderón-Zygmund operator if,*

- *there exists a kernel* $k(x,y)$ *wich satisfies the "standard estimates".*
- $< Tf, g >= \int\int k(x,y)f(y)\overline{g(x)}dxdy$, *with* $f$ *and* $g$ *two* $C^{\infty}$ *functions with compact and disjoint support.*
- $T$ *is a continuous operator from* $L^2$ *to* $L^2$.

**Theorem 7.1** *Let* $\phi$ *and* $\eta$ *two Lipschitz functions in* $\mathbb{R}^{n-1}$, *and* $B(X)$ *a* $n\times n$ *matrix, uniformly elliptic with Lipschitz coefficients, i.e there exist* $\mu$ *and* $\lambda$, *such that* $\mu|\xi|^2 \leq \langle B(X)\xi, \xi\rangle \leq \lambda|\xi|^2$, *uniformly in* $X \in \mathbb{R}^n$, *and* $|B_{i,j}(X) - B_{i,j}(Y)| \leq B'|X - Y|, \forall\ X, Y \in \mathbb{R}^n$, $\forall i, j = 1, ..., n$. *Let*

$$\mathbf{K}_{\epsilon}(B)f(x) = \int_{|x-y|>\epsilon} \frac{\eta(x) - \eta(y)}{|B(x,\phi(x))(x - y, \phi(x) - \phi(y))|^n} f(y)dy$$

*Then* $\mathbf{K}^*(B)f(x) = \sup_{\epsilon>0}|\mathbf{K}_{\epsilon}(B)f(x)|$ *is a bounded operator in* $L^2(\mathbb{R}^{n-1})$. *Moreover, there exist* $\lim_{\epsilon\to 0}\mathbf{K}_{\epsilon}(B)f(x) = \mathbf{K}(B)f(x)$ *a.e.* $x \in \mathbb{R}^{n-1}$ *and in* $L^2$, *then* $\mathbf{K}(B)$ *is a Calderón-Zygmund operator in* $\mathbb{R}^{n-1}$.

**Proof.** It is enough to consider the case when $\eta(x) = x_j$, because for any Lipschitz function $\eta(x)$, the proof follows as in the classic case (when $B = I$), [3].

Let now be

$$k(x,y) = \frac{x_j - y_j}{|B(x,\phi(x))(x - y, \phi(x) - \phi(y))|^n} \tag{7.1}$$

Taking account that the matrix $B$ has Lipschitz coefficients, this kernel satisfies the "standard estimates". Now, as in the Method of Rotations, we write

$$\mathbf{K}_{\epsilon}(B)f(x) = \frac{1}{2}\int_{|y|\geq\epsilon} k(x,x+y)f(x+y) + k(x,x-y)f(x-y)dy.$$

now, by passing to polar coordinates, with $\Sigma = \{y : |y| = 1\}$,

$$\begin{aligned} \mathbf{K}_\epsilon(B)f(x) &= \frac{1}{2}\int_\Sigma \int_{(\epsilon,\infty)} k(x, x+uy)f(x+uy) + k(x, x-uy)f(x-uy)u^{n-2}dud\sigma_y \\ &= \frac{1}{2}\int_\Sigma \mathbf{K}_{\epsilon,y}(B)f(x)d\sigma_y \end{aligned}$$

Each $x \in \mathbb{R}^{n-1}$ can be written uniquely as $x = w + ty$, where $t \in \mathbb{R}$ ,and $w \in Y$ ($Y$ denote the hyperplane orthogonal to $y$ wich passes through the origin), with this notation, we have

$$\begin{aligned} \mathbf{K}_{\epsilon,y}(B)f(x) &= \mathbf{K}_{\epsilon,y}(B)f(w+ty) = \int_{|u-t|>\epsilon} k(w+ty, w+uy)|t-u|^{n-2}f(w-uy)du \\ &= \int_{|u-t|>\epsilon} N_{w,y}(t,u)f_{w,y}(u)du \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} N_{w,y}(t,u) &= \frac{y_j}{t-u}\left|B(w+ty, \phi(w+ty))(y, \frac{\phi(w+ty)-\phi(w+uy)}{t-u}\right|^{-n} \\ &= \frac{y_j}{t-u}F_{w,t,y}\left(\frac{\phi(w+ty)-\phi(w+uy)}{t-u}\right) \end{aligned} \tag{7.3}$$

and $F_{w,t,y}(\theta) = |B(w+ty, \phi(w+ty))(y,\theta)|^{-n}$ is a $C^\infty$ function in $\theta$. furthermore, we are only interested in $\theta \in [-\|\nabla\phi\|_\infty, \|\nabla\phi\|_\infty]$, then we assume that given $\epsilon > 0$. $suppF \subset I = [-\epsilon - \|\nabla\phi\|_\infty, \epsilon + \|\nabla\phi\|_\infty]$, so $F \in L^2(I, d\theta)$(obviously by mean of convolution with an appropiate *mollifier*). Then,

$$F_{w,t,y}(\theta) = \sum_{k\in\mathbb{Z}} c_k(w,t,y)e^{ik\frac{\pi}{\epsilon+\|\nabla\phi\|_\infty}\theta}$$

with $c_k(w,t,y) = \frac{\pi}{\epsilon+\|\nabla\phi\|_\infty}\int F_{w,t,y}(\theta)e^{-ik\frac{\pi}{\epsilon+\|\nabla\phi\|_\infty}\theta}d\theta$. Since $F_{w,t,y}(\theta)$ is a smooth function, the Fourier coefficients $c_k$ are rapidily decreasing, then we have

$$\|c_k(w,t,y)\|_{L^\infty(dt)}|k|^M \le C,\ \forall\ k \text{ and } M \in \mathbb{Z}, \tag{7.4}$$

where $C = C(\|\nabla\phi\|_\infty, M, \lambda, \mu, n)$. We will later need this estimate with $M = 2N$.

Let now define

$$K(t,u) = \frac{1}{t-u} e^{i\left(\frac{\eta(t)-\eta(u)}{t-u}\right)} \quad (7.5)$$

where $\eta(t) : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, by Lemma 1 in [3], $\mathbf{K}(t,u)$ defines a continuous operator in $L^2(\mathbb{R})$, and the norm of this operator is bounded by $C(1 + \|\eta'\|_\infty)^9$, with $C$ an absolute constant. Now, (7.3) can be written as

$$N_{w,y}(t,u) = y_j \sum_{k\in\mathbb{Z}} c_k(w,t,y) K_{k,w,y}(t,u), \quad (7.6)$$

where $K_{k,w,y}(t,u)$ is a kernel as (7.5), related to the function
$\eta_{k,w,y}(t) = \frac{\pi k}{\epsilon + \|\nabla\phi\|_\infty}\phi(w+ty)$, satisfying $\|\eta'_{k,w,y}(t)\|_\infty \leq \frac{\|\nabla\phi\|_\infty |k|\pi}{\epsilon + \|\nabla\phi\|_\infty} \leq \pi|k|$.

So, there exists the operator $\mathbf{K}_{k,w,y}$ with kernel $K_{k,w,y}(t,u)$, and it is bounded in $L^2$ as follow:

$$\|\mathbf{K}_{k,w,y} f_{w,y}\|_{L^2} \leq C(1+|k|)^9 \|f_{w,y}\|_{L^2(dt)} \quad (7.7)$$

It is obviously a Calderón-Zygmund operator, then we can apply the following result:

**Theorem 7.2** **(Cotlar Lemma)** *Let $T$ be a Calderón Zygmund operator. Then, for all function $f \in L^2(\mathbb{R}^n)$ and $\forall\ x \in \mathbb{R}^n$, we have the following inequality*

$$T^*f(x) \leq C(M(Tf(x)) + M(f(x)))$$

*whith $C$ depending of $n$ and of the constants appearing in the "standard estimates" of the kernel. $M$ is the maximal Hardy-Littlewood operator, and $T^*f = \sup_{\epsilon>0}|T_\epsilon f|$.*

We can find the proof of this result, for example in [8] (p.241) .

So, we have

$$\begin{aligned} \|\mathbf{K}^*_{k,w,y} f\|_2 &\leq C(\|M\mathbf{K}_{k,w,y} f\|_2 + \|Mf\|_2) \\ &\leq C(\|\mathbf{K}_{k,w,y} f\|_2 + \|f\|_2) \quad (7.8) \\ &\leq C(1+|k|)^9 \|f\|_2 \end{aligned}$$

Now, by (7.2) and (7.6), we have

$$\begin{aligned}
\| \sup_{\epsilon > 0} |\mathbf{K}_{\epsilon,y}(B) f_{w,y}\|_{L^2(dt)} &\leq \sum_{k=-\infty}^{\infty} \|c_k(w,t,y) y_j\|_{L^\infty(dt)} \|K^*_{k,w,y} f_{w,y}(t)\|_{L^2(dt)} \\
&\leq C \sum_{k=-\infty}^{\infty} \|c_k(w,t,y)\|_{L^\infty(dt)} (1+|k|)^9 \|f_{w,y}\|_{L^2(dt)} \\
&\leq C \|f_{w,y}\|_{L^2(dt)} \sum_{k=\infty}^{\infty} |k|^{-N} \\
&\leq C \|f_{w,y}\|_{L^2(dt)}
\end{aligned}$$

using (7.4), and the estimates in (7.8).

So, by Minkowsky inequality and Fubini Theorem, we have

$$\begin{aligned}
\|\mathbf{K}^*(B) f\|_2 &= \left(\int_{\mathbb{R}^{n-1}} \sup_{\epsilon>0} |\mathbf{K}_\epsilon(B) f(x)|^2 dx\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{2}\int_{\mathbb{R}^{n-1}} \sup_{\epsilon>0} \left|\int_\Sigma \mathbf{K}_{\epsilon,y}(B) f(w+ty) dy\right|^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\int_\Sigma \|\sup_{\epsilon>0} |\mathbf{K}_{\epsilon,y}(B) f(w+ty)\|_2 dy \\
&\leq C \int_\Sigma \left(\int_Y \int_{\mathbb{R}^{n-1}} |f(w+ty)|^2 dt dw\right)^{\frac{1}{2}} dy \\
&\leq C\|f\|_2
\end{aligned}$$

with C a constant, but not always the same.

Now, we recall that $k(x,y)$ satisfies the "standard estimates", and we have just seen that the operators $\mathbf{K}_\epsilon(B)$ are uniformly bounded in $\epsilon$, so, following the classics arguments, there exists $\mathbf{K}(B)f(x) = vp\mathbf{K}_\epsilon(B)f(x)$ $\forall f \in L^2(\mathbb{R}^{n-1})$. And $\mathbf{K}$ is a Calderón - Zygmund operator. □

**Remark.** I was awarded by the referee about the preprint [7] of Marius Mitrea and Michael Taylor where a similar problem is analyzed by a different way and under differents assumptions, namely $C^1$ coefficients for the elliptic operator and considering a Lipschitz boundary domain.

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