Parallel Projected Aggregation Methods for Solving Large Inconsistent Systems. *

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Abstract

The Projected Aggregation Methods (PAM) for solving linear systems of equalities and/or inequalities, generate a new iterate x^{k+1} by projecting the current point x^k onto a separating hyperplane generated by a given linear combination of the original hyperplanes and/or halfspaces. In [9, 16, 17, 18] we introduced acceleration schemes for solving linear systems within a PAM like framework. The basic idea was to force the next iterate to belong to the convex region defined by the new separating/or aggregated hyperplane computed in the previous iteration. In this paper we extend the above mentioned methods to the problem of finding the least squares solution to inconsistent systems. In the new algorithm we used a scheme of incomplete alternate projections for minimizing the proximity function, similar to the one of Csiszár y Tusnády described in [4] which uses exact projections. The parallel simultaneous projection ACCIM algorithm in [16] is very efficient for obtaining approximations with suitable properties, and is the basis for calculating the incomplete intermediate projections. We discuss the convergence properties of the new algorithm and also present numerical experiences obtained by applying it to image reconstruction problems using the SNARK93 system [3].

Key words. Projected Aggregation Methods, Incomplete Projections, Inconsistent System.

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1 Introduction

Large and sparse systems of linear equations arise in many important applications [7, 13], as image reconstruction from projections, radiation therapy treatments planning, and in other image processing problems. A common approach to such problems is to use projection algorithms, see, e.g. Bauschke and Borwein [1], which employ projections onto convex sets in various ways, using either sequential or simultaneous schemes. In order to solve very large non-symmetric systems of linear equations, the Projected Aggregation Methods (PAM), introduced by Householder and Bauer in [14], are more efficient than those of the Cimmino or Kaczmarz type [5, 10]. They project, in every iteration, the current point onto an "aggregated" hyperplane which consists of a linear combination of the constraints of the original system. Obviously, their efficiency depends upon the way those combinations are defined, although in the consistent case the solution x^* belongs to that aggregated hyperplane. This property implies that if the current iterate x^k is projected onto that hyperplane generating x^{k+1}, then the optimal direction $d = x^* - x^{k+1}$ also belongs to it. We have developed acceleration schemes [16, 17, 18] for these sort of algorithms, based on projecting the search directions onto the aggregated hyperplanes. This idea has been applied to a variety of methods, proving that the acceleration techniques can be successfully used in conjunction with other well known algorithmic schemes. It has also been extended to the convex feasibility problem in [9] with excellent results.

Problems of image reconstruction from projections, after a suitable discretization, can be represented by a system of linear equations

$$Ax = b, \tag{1.1}$$

where A is an $m \times n$ matrix, m is the number of equations, n the size of the (unknown) image vector and b is the vector of readings. In practice, those systems are often inconsistent, and one usually seeks a point $x^* \in \Re^n$ which minimizes a certain proximity function.

In this paper we extend the above mentioned algorithms [16, 18] in order to compute the least squares solution of inconsistent systems. We derive a new simultaneous projection method, that uses a scheme of incomplete alternating projections onto convex sets for minimizing the proximity function. The ACCIM algorithm presented in our paper [16], which uses simultaneous projections in parallel, is highly efficient for obtaining the solution of consistent systems, and in particular for computing approximate projections with some desirable properties. The new algorithm can be easily parallelized and implemented for dealing with blocks of constraints. We analyze the convergence properties of the new algorithm. In the inconsistent case, it converges to the least squares solution. When the system is normalized, it converges to the minimum of the proximity function consisting of the sum of the squares of the distances to each hyperplane from the current point. We compared our algorithm with ART [5], and CAV [6], using a problem of image reconstruction from projections. The implementations were done within SNARK93 [3], a software package for testing and evaluating algorithms for image reconstruction problems.

This abridged version of the paper is organized as follows: In Section 2 we briefly review some properties of the fully simultaneous ACCIM algorithm. In section 3 the new projection ALACCIM algorithm is presented, together with the related convergence theory. In Section 4 the numerical experiences carried out within the SNARK93 are described, together with some preliminary conclusions.

2 The fully simultaneous ACCIM algorithm

Assume that the system to be solved is $Ax = b$, $A \in \Re^{m\times n}$, $m \geq n$, $b \in \Re^n$, is compatible.

From hereafter $\|x\|$ denotes 2-norm of $x \in \Re^n$, and we suppose that each row a_i of A is normalized.

We denote by x^* any solution of $Ax = b$, and I_n the identity matrix in $\Re^{n\times n}$.

We have introduced in [16] two new algorithms ACCIM and ACCAV that accelerate, in the case of consistent problems, the rates of convergence of the PPAM method in García-Palomares [10] and the CAV algorithm [6].

The version ACCIM in [16], where each block is composed by a row of the matrix, has weights w_i such that $\sum_{i=1}^m w_i = 1$ and $0 \leq w_i \leq 1$ is described by :

Algorithm 1 *(ACCIM)*

- **Initialization:** *Given* x^0, $0 < \epsilon < 1$, $k \leftarrow 0$
- **Iterative Step:** *Given* x^k, *and* Q_k *the orthogonal projector onto the orthogonal subspace of the previous direction* $\tilde{d}^{k-1}$, $k \geq 1$ *(else* $Q_0 = I_n$*),*

  Do for $i = 1, \ldots, m$ in parallel

  Compute $r_i^k = b_i - (a_i)^T x^k$.

  Define $d_i^k = r_i^k a_i$.

  Compute $\tilde{d}_i^k = Q_k(d_i^k)$.

  End do.

  Define $\tilde{d}^k = \sum_{i=1}^m w_i \tilde{d}_i^k$, and

  $x^{k+1} = x^k + \lambda_k \tilde{d}^k$, computing λ_k as given by

$$\lambda_k = argmin_\lambda \|x^k + \lambda\tilde{d}^k - x^*\|_2^2. \tag{2.2}$$

Remark 1 *In [16], the value of (2.2) is given by* $\lambda_k = \frac{\sum_{i=1}^m w_i \|d_i^k\|_2^2}{\|\tilde{d}^k\|_2^2}$.

2.1 Properties of ACCIM

With the aim of using the ACCIM algorithm as a base for defining the iterative procedure of a new method for inconsistent systems, we recall some results from [16] which will be needed:

Lemma 1 *Given* x^* *any solution of* $Ax = b$. *The sequence* $\{x^k\}$ *generated by ACCIM satisfies* *(i)* $\|x^{k+1} - x^*\|_2^2 = \|x^k - x^*\|_2^2 - \tilde{\alpha}_k$, *where*

$$\tilde{\alpha}_k = (\lambda_k)^2 \|\tilde{d}^k\|_2^2 = \|x^{k+1} - x^k\|_2^2 \text{ with } \tilde{\alpha}_k > \alpha_k, \tag{2.3}$$

being α_k the value given by PPAM method in [10] for the same weights w_i used by the ACCIM algorithm.

(ii) Given the initial point x^0, for any solution x^*, it follows that

$\|x^{k+1} - x^*\|_2^2 = \|x^0 - x^*\|_2^2 - \mathcal{S}_k$, *where*

$\mathcal{S}_k = \sum_{j=1}^{j=k} \tilde{\alpha}_j = \sum_{j=0}^{j=k} \|x^{j+1} - x^j\|_2^2$.

(iii) Given x^0 initial, the sequence $\{x^k\}$ of ACCIM converges to

$$x^*_{min} = argmin\{\|x^* - x^0\|^2, x^* \in \Re^n : Ax^* = b\}. \tag{2.4}$$

Proof. The results correspond to the theory developed in [16], Lemmas 2.7- 2.9, and are characteristic of the PAM methodology except in the differences between the value of $\tilde{\alpha}_k$ and the value given by the more general PPAM method described in [10]. □

The version ACCAV in [16], where each block is composed by a row of the matrix, is identical to ACCIM except by the weights $w_i > 0$, which are taken as equal to those of the CAV algorithm [6]. Denoting by s_j the number of non zero elements in the $j-th$ column of the matrix A, and by a_{i_j} the element j of row i, the weights are calculated as $w_i = \frac{1}{\sum_{j=1}^{n} s_j (a_{i_j})^2}$.

Remark 2 *The new weights w_i are constant with respect to k as those considered in the hypotheses of Lemma 2.7 given in [16]. In that paper a result was proved showing that the sequence generated by the ACCAV speeds up the convergence of the algorithm CAV, when the system $Ax = b$ is consistent.*

3 Inconsistent case.

It is known that there are some applications [5, 7, 13] which need to solve a possibly inconsistent system $Ax = b$, $A \in \Re^{m \times n}$, $b \in \Re^m$, $m > n$. So, it is necessary to consider the standard problem:

$$\min_{x \in \Re^n} \|b - Ax\|_W^2 \tag{3.5}$$

whose solution coincides with the one of the problem:

$$A^T W A x = A^T W b.$$

We will assume that $rank(A) = n$ and the diagonal matrix of weights W is nonsingular. Hence, $A^T A$ and $A^T W A$ are nonsingular matrices.

The particular acceleration scheme described in [16], for the ACCIM and ACCAV algorithms, can be extended to the inconsistent case if in the computation of λ_k, the 2-*norm* is replaced by $\|..\|_{A^TWA}$, considering

$$\lambda_k = argmin_\lambda \|x^k + \lambda \tilde{d}^k - x^*\|^2_{A^TWA}, \tag{3.6}$$

$d^k = \sum_{i=1}^m r_i^k a_i$, like in ACCIM, and $\tilde{d}^k = P_{v^\perp}(d^k)$, with $v = A^T W A \tilde{d}^{k-1}$.

In this paper we replace the use of the above mentioned weighted norm by the Euclidean norm within an appropriate context.

3.1 The Alternating Minimization Algorithm

Our idea was to apply an alternate scheme, similar to the Csiszár and Tusnády method [8], for deriving a new method for solving the problem (3.5) based on ACCIM for obtaining incomplete projections onto a convex set.

The simplified version of the Csiszár and Tusnády algorithm described in [4] considers, $\mathcal{P}$ and $\mathcal{Q}$, which are two convex sets in the n-dimensional Euclidean space $\Re^n$. Let $\Theta(p,q)$ be a real-valued function defined for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$:

Algorithm 2

Iterative step *: Given q^k, find p^{k+1} as:*

$$p^{k+1} = argmin\{\Theta(p, q^k) | p \in \mathcal{P}\}$$

then calculate q^{k+1} by solving

$$q^{k+1} = argmin\{\Theta(p^{k+1}, q) \mid q \in \mathcal{Q}\}.$$

End.

Remark 3 *In [4] it is proved that $\Theta(p^k, q^k)$ is a decreasing sequence. In order to use the general function $\Theta(p,q)$, for proving convergence to the points p^* and q^* which minimize $\Theta(p,q)$ for all p and q of $\mathcal{P}$ and $\mathcal{Q}$ respectively, it is necessary to impose some conditions. In particular, when $\Theta(p,q) = ||p-q||^2$ the convergence of the Algorithm 2 is proved.*

Our proposal is to replace the projection onto the first set by either an approximate or an incomplete one. For the second set, we keep the idea of using an exact projection without increasing the computational cost.

3.2 Incomplete Alternating Projections Algorithm

Given $Ax = b$, $A \in \Re^{m\times n}$, $b \in \Re^m$, $m \geq n$, $rank(A) = n$, we consider the augmented system

$$Ax - r = b, \quad r \in \Re^m, \quad x \in \Re^n. \tag{3.7}$$

Denoting by (x^*, r^*), $x^* \in \Re^n$, $r^* \in \Re^m$, the solution of

$$min\{||r||^2 : r = Ax - b, \quad x \in \Re^n\}. \tag{3.8}$$

We define, as in Algorithm 2, two convex sets in the $(n+m)$-dimensional Euclidean space $\Re^{n+m}$

$$\mathcal{P} = \{(x,r) \in \Re^{n+m} : x \in \Re^n, r \in \Re^m, Ax - r = b\}$$

and

$$\mathcal{Q} = \{(x,0) : x \in \Re^n, 0 \in \Re^m\}$$

adopting $\Theta(p,q) = \|p-q\|^2$, for all p and q of $\mathcal{P}$ and $\mathcal{Q}$ respectively.

Given $p^k \in \mathcal{P}$, and its projection q^k onto $\mathcal{Q}$, we denote $p_{min}(q^k)$ the solution of the problem

$$\min\{\|p-q^k\| : \quad p \in \mathcal{P}\}. \tag{3.9}$$

In particular, if we denote $p^* = (x^*, r^*)$, and $q^* = (x^*, 0)$, by definition of (x^*, r^*) in (3.8), we get $p^* = p_{min}(q^*)$.

In the new algorithm, given $q^k \in \mathcal{Q}$ instead of defining $p^{k+1} = p_{min}(q^k)$ as it was done in Algorithm 2, we use $p^{k+1} = p_a^{k+1}$, where $p_a^{k+1} \in \mathcal{P}$, is a point obtained by means of the incomplete resolution of the problem (3.9). We now formulate the definition of p_a^{k+1}.

Aiming at obtaining properties of the sequence $\{p^k\}$ generated by the new algorithm which guarantees convergence to the solution of (3.8) we establish an "acceptance condition" which an approximation $\breve{p} = (z,\mu)$ of $p_{min}(q^k)$ must satisfy.

Definition 1 *An approximation $\breve{p} = (z,\mu)$ of $p_{min}(q^k)$ is acceptable if*

$$\|\breve{p} - p_{min}(q^k)\|_2^2 \leq \gamma \|\breve{p} - p^k\|_2^2, \quad \textit{with} \;\; 0 < \gamma < 1. \tag{3.10}$$

Remark 4 *If $p_{min}(q^k) \neq p^k$, there exists $\breve{p} = q^k + \alpha(p_{min}(q^k) - q^k)$ such that $\|\breve{p} - p_{min}(q^k)\|_2^2 \leq \gamma_2 \|p^k - p_{min}(q^k)\|_2^2$, for a particular $0 < \gamma_2 < 1$. Those $\breve{p}$ also satisfy $\|\breve{p} - p^k\|_2^2 = \|\breve{p} - p_{min}(q^k)\|_2^2 + \|p^k - p_{min}(q^k)\|_2^2$, therefore $\|\breve{p} - p^k\|_2^2 \geq (1 + 1/\gamma_2)\|\breve{p} - p_{min}(q^k)\|_2^2$. Then, $\breve{p}$ exists satisfying condition (3.10), $\|\breve{p} - p_{min}(q^k)\|_2^2 \leq \gamma_2/(1+\gamma_2)\|\breve{p} - p^k\|_2^2$, for $\gamma = \gamma_2/(1+\gamma_2)$. Furthermore, if $p_{min}(q^k) = p^k$, the unique $\breve{p}$ which satisfies (3.10) is $p_{min}(q^k)$.*

Since (3.10) is not practical because we do not know $p_{min}(q^k)$, we consider the following definition.

Definition 2 *Given an approximation $\breve{p} = (z,\mu)$, $z \in \Re^n$, $\mu \in \Re^m$, of $p_{min}(q^k)$, we denote by $P(\breve{p}) = (z, \mu + s)$, the solution of the system $Ax - r = b$ which satisfies $\mu + s = Az - b$.*

Now we can establish the sort of condition we were looking for:

Practical condition: An approximation $\breve{p} = (z,\mu)$ of $p_{min}(q^k)$ is acceptable if it satisfies

$$\|\breve{p} - P(\breve{p})\|_2^2 \leq \gamma \|\breve{p} - p^k\|_2^2, \quad \text{with} \;\; 0 < \gamma < 1. \tag{3.11}$$

Remark 5 *In particular, $\breve{p} = p_{min}(q^k)$ satisfies the practical condition. In order to see that it is feasible to find an approximation $\breve{p}$ of $p_{min}(q^k)$, by an iterative algorithm, satisfying (3.11) it is necessary to consider its convergence properties.*

We will use the ACCIM algorithm and, taking into account its convergence properties, we will prove (3.11).

The efficiency for obtaining $\breve{p}$, satisfying (3.11), depends on the algorithm used for solving (3.9).

Lemma 2 *Given $p^k = (x^k, r^k)$, $q^k = (x^k, 0)$. Under the assumptions of Lemma 1, if the sequence $\{(z^j, \mu^j)\}$ generated by an iterative algorithm, with $(z^0, \mu^0) = q^k$, converges to $p_{min}(q^k)$, then*

(i) If $p_{min}(q^k) \neq p^k$, an index $\tilde{j}$ exists such that $(z^{\tilde{j}}, \mu^{\tilde{j}})$ satisfies (3.11).

(ii) If $p_{min}(q^k) = p^k$, then the unique solution of (3.11) is p^k.

Proof. Assuming that $\|p^k - p_{min}(q^k)\| \neq 0$, and if for all $j \geq 0$ the approximation $\breve{p} = (z^j, \mu^j)$ does not satisfy (3.11), then

$$\|\breve{p} - P(\breve{p})\|_2^2 > \gamma \|\breve{p} - p^k\|_2^2, \quad \text{for all} \quad j \geq 0. \tag{3.12}$$

If we denote $s^j = Az^j - \mu^j - b$, due to our hypothesis, $\breve{p}$ goes to $p_{min}(q^k)$ and also we know that $\|s^j\|$ goes to 0. Furthermore, from the definition $P(\breve{p})$ we get that $\|\breve{p} - P(\breve{p})\|_2^2 = \|s^j\|^2$, and by (iii) of Lemma 1 the successive $\breve{p}$ satisfy $\|\breve{p} - p^k\|_2^2 = \|\breve{p} - p_{min}(q^k)\|_2^2 + \|p^k_{min} - p^k\|_2^2$. Therefore, to assume that the inequality (3.12) leads to $\|s^j\|_2^2 > \gamma \|p_{min}(q^k) - p^k\|_2^2$, for all $j \geq 0$. Since s^j tends to zero, we get $\|p_{min}(q^k) - p^k\|_2^2 = 0$. Therefore, since the hypothesis $\|p_{min}(q^k) - p^k\|_2^2 \neq 0$, an index $\tilde{j}$ must exists for which $\breve{p} = (z^{\tilde{j}}, \mu^{\tilde{j}})$, satisfying the condition (3.11).

When $p_{min}(q^k) = p^k$, as a consequence of (iii) of Lemma 1, we get that $\|\breve{p} - p^k\|_2^2 \leq \|\breve{p} - P(\breve{p})\|_2^2$, for all $\breve{p} = (z^j, \mu^j)$. Therefore, el unique vector satisfying (3.11) is $\breve{p} = p_{min}(q^k) = p^k$. □

Remark 6 *In the second case of the previous Lemma, if we use ACCIM for computing the sequence $\{(z^j, \mu^j)\}$, with $(z^0, \mu^0) = q^k$, then $(z^1, \mu^1) = p^k = p_{min}(q^k)$, as a consequence of the iterative step.*

In the new algorithm we define $p^{k+1} = p_a^{k+1} = P(\breve{p})$, if $\breve{p}$ satisfies (3.11).

Then as in Algorithm 2, once $p^{k+1} = p_a^{k+1}$ is defined, the second step of the new alternate projection algorithm computes

$$q^{k+1} = argmin\{\|p^{k+1} - q\| \ : \quad q \in \mathcal{Q}\}.$$

It follows immediately that this second step does not require additional computations due to the particular definition of the set $\mathcal{Q}$.

Therefore, the scheme iterative of the new algorithm is the following:

Algorithm 3

Iterative step *: Given $p^k = (x^k, r^k) \in \mathcal{P}$, and $q^k = (x^k, 0) \in \mathcal{Q}$, $x^k \in \Re^n$, $r^k \in \Re^m$,*

- *find $p^{k+1} \in \mathcal{P}$, $p^{k+1} = (x^{k+1}, r^{k+1})$ such that $x^{k+1} = z$, $r^{k+1} = \mu + s$, being $\breve{p} = (z, \mu)$ an approximation of solution of (3.9) satisfying the condition (3.11), and $P(\breve{p}) = (z, \mu + s)$, then*
- *define q^{k+1}*

  $q^{k+1} = (x^{k+1}, 0)$.

End.

We describe in the following a practical algorithm which uses ACCIM for obtaining $\breve{p} \approx p_{min}(q^k)$ satisfying (3.11).

Algorithm 4 *(ALACCIM)*

- **Initialization:** *Given* $p^0 = (x^0, r^0)$, $r^0 = Ax^0 - b$, $0 < \gamma < 1$.
  Let $q^0 = (x^0, 0)$ *be,* $k \leftarrow 0$.
- **Iterative Step:** *Given* $p^k = (x^k, r^k)$ *and* $q^k = (x^k, 0)$.
  Calculate $\breve{p}$, approximation of $p_{min}(q^k)$ satisfying (3.11), applying ACCIM as follows:
  Define $y^0 = (z^0, \mu^0) = q^k$ *the initial point. For solving* $Ax - r = b$, *iterate until finding* $y^j = (z^j, \mu^j)$ *such that* $s^j = Az^j - \mu^j - b$ *satisfies (*3.11*), that is*

$$\|s^j\|^2 \leq \gamma(\|r^k\|^2 - \mathcal{S}_j), \;\; with \quad \mathcal{S}_j = \sum_{i=1}^{j} \|y^i - y^{i-1}\|^2.$$

  Define $p^{k+1} = (z^j, \mu^j + s^j)$.
  Define $q^{k+1} = (z^j, 0)$.
  $k \leftarrow k + 1$.

Lemma 3 *Let* $p^* = (x^*, r^*)$ *and* $q^* = (x^*, 0)$ *be the points which minimize* $\|p - q\|$ *for all* $p \in \mathcal{P}$ *and* $q \in \mathcal{Q}$. *If* $\{p^k\} = \{(x^k, r^k)\}$ *is generated by the Algorithm* 4, *then*

(i) $\|r^{k+1}\|^2 \leq \|r^k\|^2 - (1 - \gamma)\|p^k - p_{min}(q^k)\|^2$, *being* $p_{min}(q^k)$ *defined in (*3.9*).*

(ii) The sequence $\{\|r^k\|\}$ *is decreasing and bounded by* $\|r^*\|$, *therefore it converges.*

(iii) The sequence $\{\|p^k - p_{min}(q^k)\|^2\}$ *goes to zero.*

(iv) The sequence $\{\|p^{k+1} - p_{min}(q^k)\|^2\}$ *goes to zero.*

(v) The sequence $\{\|p^{k+1} - p^k\|^2\}$ *goes to zero.*

(vi) The sequence $\{\|q^{k+1} - q^k\|^2\}$ *goes to zero.*

(vii) The sequence $\{\|A^T r^k\|\}$ *tends to zero.*

Furthermore, assuming that $rank(A) = n$,

(viii) $\{p^k\} = \{(x^k, r^k)\}$ *converges to* p^*, *and* $\{q^k\} = \{(x^k, 0)\}$ *converges to* q^*.

Proof. We get, as a consequence of the definition of q^{k+1}, that $\|r^{k+1}\|^2 \leq \|p^{k+1} - q^k\|^2$. Furthermore, due to (iii) of Lemma 1, $\|q^k - p^{k+1}\|^2 = \|\breve{p} - p^{k+1}\|^2 + \mathcal{S}_j$. Due to the definition of $\breve{p}$, $\|q^k - p^{k+1}\|^2 \leq \gamma\|\breve{p} - p^k\|^2 + \mathcal{S}_j$. Since $\mathcal{S}_j = \|r^k\|^2 - \|\breve{p} - p^k\|^2$, we get $\|q^k - p^{k+1}\|^2 \leq \gamma\|\breve{p} - p^k\|^2 + \|r^k\|^2 - \|\breve{p} - p^k\|^2$. Then $\|q^k - p^{k+1}\|^2 \leq \|r^k\|^2 - (1-\gamma)\|\breve{p} - p^k\|^2$.

Considering $\|\breve{p} - p^k\|^2 = \|\breve{p} - p_{min}(q^k)\|^2 + \|p_{min}(q^k) - p^k\|^2$, we get $\|r^{k+1}\|^2 \leq \|r^k\|^2 - (1-\gamma)\|p^k - p_{min}(q^k)\|^2$.

We obtain (ii) as a consequence of (i) and the definition of r^*. Also (iii) follows from (ii).

To prove (iv), since $\|p^k - \breve{p}\|^2 \geq 1/\gamma\|\breve{p} - p^{k+1}\|^2$, considering that $\|\breve{p} - p^{k+1}\|^2 = \|\breve{p} - p_{min}(q^k)\|^2 + \|p^{k+1} - p_{min}(q^k)\|^2$, and $\|p^k - \breve{p}\|^2 = \|\breve{p} - p_{min}(q^k)\|^2 + \|p^k - p_{min}(q^k)\|^2$, then $\|p^k - \breve{p}\|^2 \geq 1/\gamma(\|\breve{p} - p_{min}(q^k)\|^2 + \|p^{k+1} - p_{min}(q^k)\|^2)$. Therefore, $\|p^k - p_{min}(q^k)\|^2 \geq \frac{(1-\gamma)}{\gamma}\|\breve{p} - p_{min}(q^k)\|^2 + 1/\gamma\|p^{k+1} - p_{min}(q^k)\|^2$. It follows from (iii) $\|p^{k+1} - p_{min}(q^k)\|^2$ tends to zero.

Since $\|p^{k+1} - p^k\| \leq \|p^{k+1} - p_{min}(q^k)\| + \|p_{min}(q^k) - p^k\|$ and the right-hand sum tends to zero because of (iii) and(iv), then (v) follows. Also (vi) is a consequence of (v).

From (iii), we know that $\{\|p^k - p_{min}(q^k)\|^2\}$ goes to zero. Using the definition of $p_{min}(q^k)$, (vii) follows.

Finally, taking into account the hypothesis that $rank(A) = n$, the previous results, and applying Theorem 14.1.4 in J.M. Ortega and W. C. Rheinboldt [15], we prove (viii). □

If in the Algorithm 4 we replace for obtaining $\breve{p}$, the algorithm ACCIM by ACCAV [16] which uses w_i of CAV, we get the version ALACCAV.

4 Experimental Results on an Image Reconstruction Problem from Projections

We compared our algorithm using an image reconstruction problem from projections. The main algorithms for comparison were ART [5] and CAV [6]. All methods were implemented sequentially. The implementations were done within SNARK93, a software package for testing and evaluating algorithms for image reconstruction [3]. The experiences were run on a PC Pentium III, 800MHz, with 256 Mb Ram and 128 Mb Swap. The algorithms were compared on the basis of their qualitative and quantitative behavior. We show the performance of the algorithms on the reconstruction of the Herman head phantom (Herman [12]), defined by a set of ellipses, with a specific attenuation value attached to each original elliptical regions. The systems (see Censor et al. [6] for a more complete description) are basically inconsistent, because in the fully discretized model each line integral of the attenuation along the i-th ray is approximated by a finite sum. This matches the real-life situation where the right-hand-side ($b_i's$) of the systems are actual X-ray readings through an object but the region of interest is discretized. The performance of the new ALACCIM algorithm is compared with CAV and ART. We use the term iteration (Iter) to mean a single whole sweep through all equations of the system. In the ALACCIM algorithm the definition of a new iterate (major iteration) requires one or more of the above mentioned Iter, two as an average at the beginning, and four or five when close to the solution. After running ALACCIM we run all other algorithms, extracting and comparing the results corresponding to the same number of iterations. All our experiments were initiated with $x^0 = 0$.

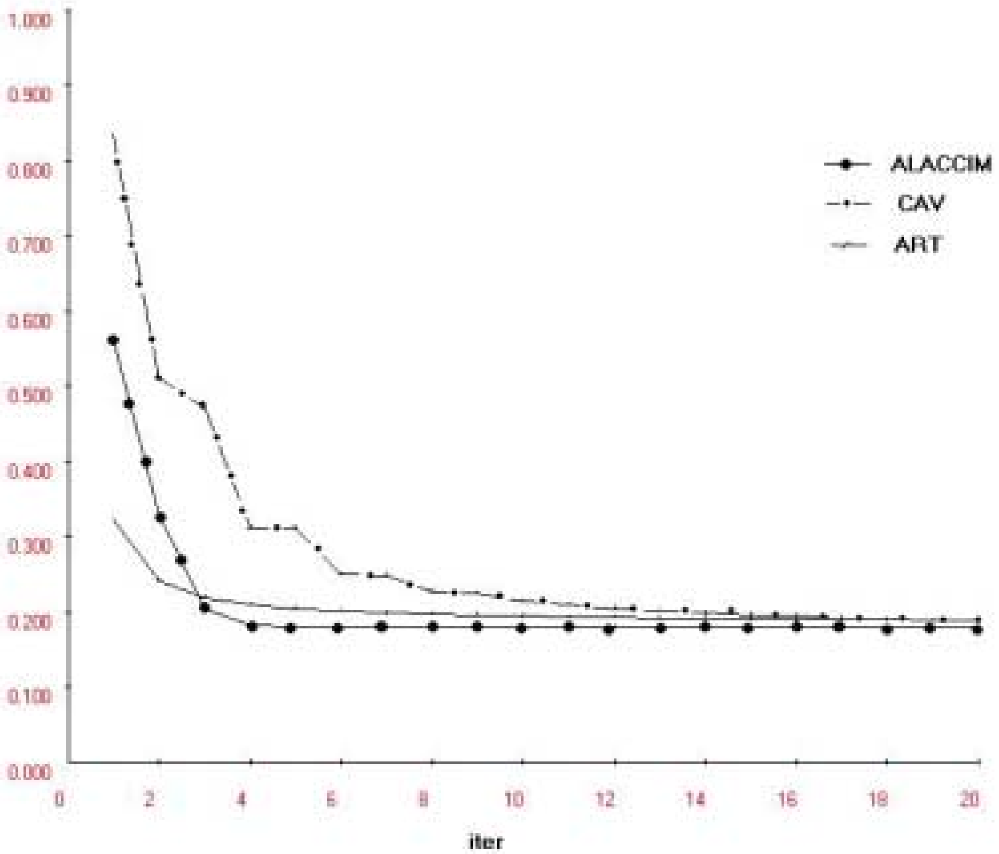
	Equations	Variables	Image Size	Projections	Rays
case 1	13137	13225	115×115	151	87
case 2	26425	13225	115×115	151	175
case 3	126655	119025	345×345	365	347
case 4	232275	119025	345×345	475	489

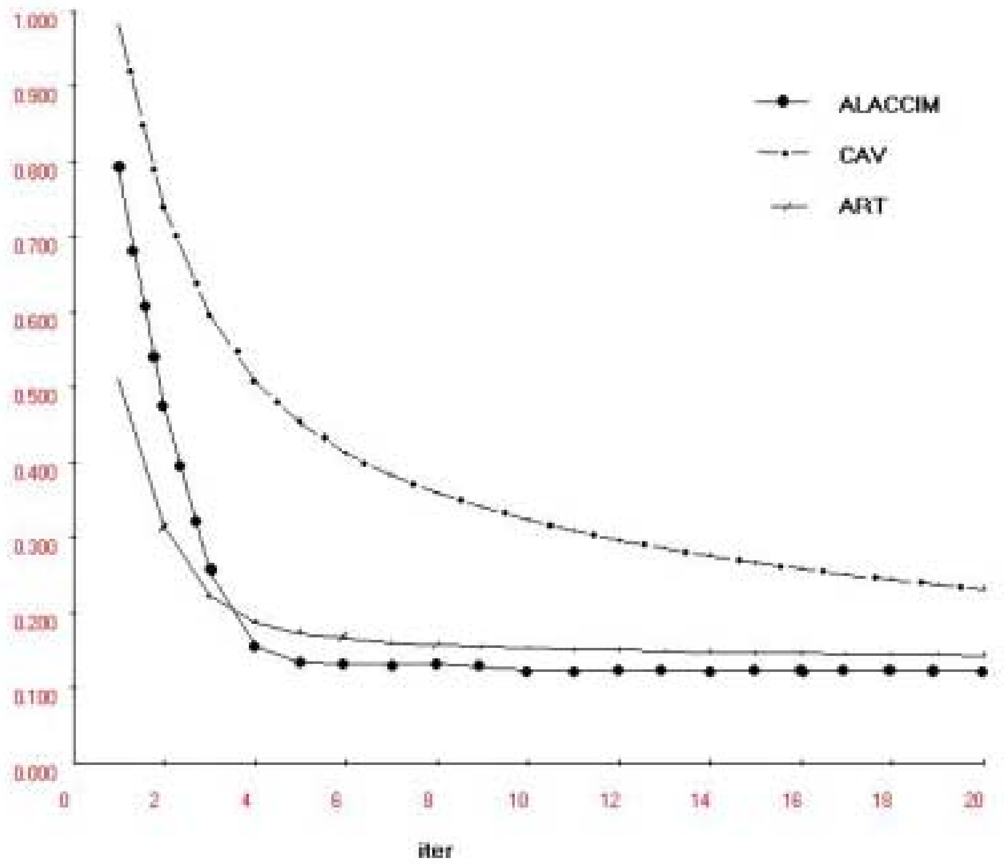
Table 4.1 : Test cases

4.1 Comparisons

Four different test cases, using the Herman head phantom, were run. Those tests are the same appearing in [6], which arise from considering a different number of projections and number of rays per projection, leading in such a way to systems with a different number of variables and equations, as shown in Table 4.1.

As an example of numerical experiences we show graphs that compare the three algorithms in the cases 1-3. We can observe a very efficient behaviour of ALACCIM because in a few itera





4.2 Conclusions

The acceleration scheme applied in the ACCIM algorithm is the basis for extending its applicability to other class of algorithms suitable for parallel processing. In particular, we used the same approach in the more general ALACCIM, for computing incomplete projections. This algorithm can be easily implemented in parallel as the CAV method, and can be extended for considering blocks of equalities in a similar way as it was done for the ACIPA algorithm in [9] for linear inequalities. That extension will be presented in a forthcoming paper, together with the results of its application to image reconstruction problems.

The general acceleration scheme within the PAM framework adapted for dealing with inconsistent systems presented in this paper, turned out to be very efficient when applied to different projection algorithms. The practical consequences of the new technique are far reaching for real-world reconstruction problems in which iterative algorithms are used, and in other fields where there is a need to solve large and sparse unstructured systems of linear equations.

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